

The harmonic oscillator on Riemannian and Lorentzian configuration spaces of constant curvature

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Abstract

The harmonic oscillator as a distinguished dynamical system can be defined not only on the Euclidean plane but also on the sphere and on the hyperbolic plane, and more generally on any configuration space with constant curvature and with a metric of any signature, either Riemannian (definite positive) or Lorentzian (indefinite). In this paper we study the main properties of these ‘curved’ harmonic oscillators simultaneously on any such configuration space, using a Cayley-Klein (CK) type approach, with two free parameters κ_1, κ_2 which altogether correspond to the possible values for curvature and signature type: the generic Riemannian and Lorentzian spaces of constant curvature (sphere \mathbf{S}^2 , hyperbolic plane \mathbf{H}^2 , AntiDeSitter sphere \mathbf{AdS}^{1+1} and DeSitter sphere \mathbf{dS}^{1+1}) appear in this family, with the Euclidean and Minkowski spaces as flat limits.

We solve the equations of motion for the ‘curved’ harmonic oscillator and obtain explicit expressions for the orbits by using three different methods: first by direct integration, second by obtaining the general CK version of the Binet’s equation and third, as a consequence of its superintegrable character. The orbits are conics with centre at the potential origin in any CK space, thereby extending this well known Euclidean property to any constant curvature configuration space. The final part of the article, that has a more geometric character, presents those results of the theory of conics on spaces of constant curvature which are pertinent.

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The harmonic oscillator. Integrability and Superintegrability. Spaces of constant curvature. Central potentials. Conics on the sphere and hyperbolic plane. Conics in Lorentzian spaces.

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1 Introduction

In a sense this article can be considered as a sequel or continuation of a previous paper [12] which was devoted to the study of mechanical systems on Riemannian configuration spaces with constant curvature $\kappa \neq 0$. Geodesic motion, the theory of symmetries and general results on central potentials were discussed in the first part of [12], while in the second part attention was focused on the Kepler problem in \mathbf{S}^2 and \mathbf{H}^2 . Now, we present a similar analysis for the harmonic oscillator, yet extending the scope so as to include also the much less explored cases where the configuration space is a *Lorentzian* manifold with constant curvature. We follow the approach of [12], which contains the fundamental ideas and motivations, and also use the notation, ideas and results discussed in [50, 51].

The study of mechanical systems on Riemannian spaces has been mainly done in connection with relativity and gravitation. Nevertheless, before relativity, the study of both Kepler and harmonic oscillator potentials in spaces of *constant curvature* had also been done from the viewpoint of classical nonrelativistic mechanics (see [17] for an historical account of the research made until the first years of the XX century and references in [12] for more recent papers including also some quantum problems on spaces with curvature as, e.g., the hydrogen atom in a spherical or hyperbolic geometry). It is interesting to point out that [39], a book on geometry, includes however a final chapter devoted to mechanics (the title of this chapter was “*Nichteuklidische Mechanik*” in the original edition but was changed to “*Mechanik und spezielle Relativitätstheorie*” in the revised 1923 edition); in addition to rather general properties linking geometry with mechanics, this chapter contains the basics of a study of the harmonic oscillator in constant curvature; polar coordinates are used and the approach is basically Newtonian.

It is well known that the Kepler problem and the harmonic oscillator are the two more important superintegrable systems in Euclidean space (see for instance the recent book [16]), and, as it was to be expected, they have ‘curved versions’ which remain superintegrable in spherical or hyperbolic configuration spaces [57, 61]. This known property implicitly underlies some classical papers as [56] or [28]. On the other hand, much work has been recently done in the study of superintegrable systems in non-Euclidean spaces [4, 5, 6, 9, 10, 20, 30, 32, 34, 35, 47, 53, 57, 60] and this fact has intensified the interest for the study of the ‘curved’ versions of these two systems and their relations [31, 33, 49, 52]. A further step, which we take in this paper, is to extend these studies also to the case where the configuration space is itself a constant curvature Lorentzian manifold. This case was definitely not taken into account in the previous papers, and opens some views into a relatively unknown field. For some work related to dynamics in Lorentzian manifolds, see [7, 42, 43, 59].

The three *classical* spaces with constant curvature κ , to wit, the sphere \mathbf{S}_κ^2 with $\kappa > 0$, Euclidean plane \mathbf{E}^2 for $\kappa = 0$ and hyperbolic plane \mathbf{H}_κ^2 for $\kappa < 0$, can be considered as the three different instances in the family of homogeneous Riemannian manifolds $V_\kappa^2 = (\mathbf{S}_\kappa^2, \mathbf{E}^2, \mathbf{H}_\kappa^2)$. A technique for considering these three spaces at the same time in a unique family, with the curvature κ as a parameter $\kappa \in \mathbb{R}$ was first introduced by Weierstrass and Killing [17] and lies at the origin of the so-called Weierstrass model for the hyperbolic plane; for some reason modern presentations usually restrict to the standard value $\kappa = -1$, thus losing from direct view how some properties depend on the curvature. If κ is left explicitly, this allows consideration of the ‘curved’ harmonic oscillator (or Kepler problem) on constant curvature spaces as arising from Lagrangians depending on κ as a parameter and understood as defined in a generic space V_κ^2 (either $(\mathbf{S}_\kappa^2, \mathbf{E}^2, \mathbf{H}_\kappa^2)$) by using a unique κ -dependent expression. The ‘curved’ systems appear as a ‘ κ -deformation’ of the well known Euclidean system and they can be defined without ambiguities because superintegrability picks up essentially a unique ‘ κ -deformation’ among the many (non-superintegrable) potentials having the (Euclidean) Kepler or oscillator as its ‘ $\kappa \rightarrow 0$ limit’.

A convenient tool for the use of κ as a parameter are the following κ -trigonometric ‘Sine’ and ‘Cosine’ functions

$$C_\kappa(x) := \begin{cases} \cos \sqrt{\kappa} x \\ 1 \\ \cosh \sqrt{-\kappa} x \end{cases}, \quad S_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x \\ x \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x \end{cases} \quad \begin{matrix} \kappa > 0 \\ \kappa = 0 \\ \kappa < 0 \end{matrix}, \quad (1)$$

as well as the ‘Tangent’ $T_\kappa(x) = S_\kappa(x)/C_\kappa(x)$. These functions allow us to write κ -dependent expressions in a unified way for the whole previous family of spaces V_κ^2 so that the computations, statements and results are unified too.

This approach, used in some previous papers [3, 12, 47, 50, 51], has a potentiality beyond the unification for the three constant curvature configuration spaces V_κ^2 : the mathematically natural frame for the ‘ κ as a parameter’ idea [11, 26, 27, 55] involves not just one single parameter κ , but *two parameters*, κ_1 and κ_2 , which correspond to a space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ with constant curvature κ_1 and whose metric is definite (Riemannian), degenerate or indefinite (Lorentzian) according to the κ_2 sign (see e.g. [2, 13, 24, 25]).

In more detail, the plan of this article is as follows: In Sec. II we analyse, in a single run, the free geodesic motion on the three Riemannian spaces $V_\kappa^2 \equiv (\mathbf{S}_\kappa^2, \mathbf{E}^2, \mathbf{H}_\kappa^2)$, and in the three Lorentzian spaces $L_\kappa^2 \equiv (\mathbf{AdS}_\kappa^{1+1}, \mathbf{M}^{1+1}, \mathbf{dS}_\kappa^{1+1})$ as well as the Killing vector fields in the general $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ and its correspondent Noether symmetries. We have divided this section in two subsections, using respectively geodesic polar coordinates and geodesic parallel coordinates.

Sec. III, that can be considered as the central part of the article, is devoted to the study of the harmonic oscillator on the general space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$; the specialization to $\kappa_1 = \kappa$, $\kappa_2 = 1$ affords the relevant results for the harmonic oscillator in the three ‘classical’ Riemannian spaces of constant curvature $(\mathbf{S}_\kappa^2, \mathbf{E}^2, \mathbf{H}_\kappa^2)$, and further specialization to $\kappa = 0$ leads to the harmonic oscillator in \mathbf{E}^2 . The results also cover the harmonic oscillator on Lorentzian configuration spaces.

After an introduction, we have divided this section into three subsections: in the first part we consider the equivalent one-dimensional problem and we draw some information and a classification of the orbits previous to obtaining any closed-form solution for the motion; in the second part we solve explicitly the problem and we obtain closed expressions of the orbits. The results can be obtained by using three different methods: first by direct integration, second by obtaining the $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ -dependent version of the Binet’s equation, and third by exploiting anew the superintegrability of the problem; this last method is closer to the one usually employed in the Euclidean oscillator, which leans on its separable character in Cartesian coordinates, a property which is not shared by the Kepler potential. The third subsection is devoted to the analysis and classification of orbits, with the emphasis restricted to the three ‘classical’ Riemannian configuration spaces with nonzero curvature κ . This third subsection also interprets the trajectories obtained above as conics in curved spaces; of course all the results reduce to well known Euclidean trajectories —ellipses centred in the potential origin—, when we specialize the parameters to their standard Euclidean values $\kappa_1 = 0, \kappa_2 = 1$. Our description of conics goes beyond the classical papers on this topics published around 1900 (where the emphasis was mainly projective, although some metrical aspects are also discussed; see, e.g., [58] and references therein or [15, 36]). And, at any rate, conics in locally Minkowskian were definitely not considered at all in any of these works; probably [8] is the first paper dealing with this subject.

Sec. IV, that has a more geometric character, gives some information on the theory of conics in the general CK space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$, leading to the identification of the harmonic oscillator orbits with conics for any value of the parameters κ_1, κ_2 (this is, either for Riemannian as well as for Lorentzian configuration spaces), and serves as a geometrical counterpart and complement to the information

already provided on Sec. III. Additional details are provided mainly in the Riemannian case; we plan to discuss the case of a Lorentzian configuration space in more extension in a forthcoming paper. Finally, in Sec. V we make some final comments.

We mention that most results here also hold (in a suitably reformulated way) for the n -dimensional version of the harmonic oscillator, which is well known to be superintegrable [26, 40].

2 Dynamics on $\mathcal{S}_{\kappa_1[\kappa_2]}^2$: Geodesic motion, Noether symmetries and constants of motion

We start by discussing some details on the motion of a particle in a configuration space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ in the general CK case, where the parameters κ_1, κ_2 may have any real value. This (κ_1, κ_2) -dependent formalism contains *nine* essentially different Cayley-Klein spaces, because by scaling the units of length by λ and of angle by α , the values of κ_1 and κ_2 transform as $\kappa_1 \rightarrow \lambda^2 \kappa_1$, $\kappa_2 \rightarrow \alpha^2 \kappa_2$. Then without any loss of generality, any CK space can be brought to its *standard form*, with either $\kappa_1 = 1, 0, -1$ and $\kappa_2 = 1, 0, -1$. The standard form of any expression in the CK formalism coincides with the result one would obtain by working from the outset in a single space. Should we proceed this way, however, the consideration of how details change when there is a variation of curvature or signature type would require an additional separate study. The distinctive trait in the CK formalism is that the dependence on κ_1 and κ_2 is built-in, and makes a further study of the limiting processes fully redundant. For this reason we will keep the general form, with explicit κ_1 and κ_2 in most of the paper, stressing when required the specific properties holding after a specialization for the values of κ_1, κ_2 .

The presence of *two parameters* in the CK family of two-dimensional spaces is related to the Cayley-Klein theory of projective metrics, and underlies the length/angle duality which is the residue of the general duality in projective geometry when projective metrics are taken into account. It is also related to the existence of two commuting involutions in their isometry Lie algebras [21]

Within the generic *standard* choices $\kappa_2 = \pm 1$ for the signature type, the CK family $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ with $\kappa_2 = 1$ includes the three ‘classical’ Riemannian spaces with constant curvature $\kappa_1 = \kappa$ and the CK family $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ with $\kappa_2 = -1$ includes the three Lorentzian spaces with constant curvature (kinematically interpretable as homogeneous space-times). In the non-generic case $\kappa_2 = 0$ the CK spaces can be interpreted as the three 1+1 non-relativistic space-times, which are limits of the spaces with $\kappa_2 \neq 0$; see [23, 44] and references therein. These nine spaces can be conveniently displayed in a Table; for more details see [23, 24, 25].

Table 1: The nine standard two-dimensional CK spaces $\mathcal{S}_{\kappa_1[\kappa_2]}^2$.

Measure of angle & Sign of κ_2	Measure of distance & Sign of κ_1		
	Elliptic $\kappa_1 = 1$	Parabolic $\kappa_1 = 0$	hyperbolic $\kappa_1 = -1$
Elliptic $\kappa_2 = 1$	Elliptic \mathbf{S}^2	Euclidean \mathbf{E}^2	hyperbolic \mathbf{H}^2
Parabolic $\kappa_2 = 0$	Co-Euclidean Oscillating NH \mathbf{ANH}^{1+1}	Galilean \mathbf{G}^{1+1}	Co-Minkowskian Expanding NH \mathbf{NH}^{1+1}
hyperbolic $\kappa_2 = -1$	Co-hyperbolic Anti-de Sitter \mathbf{AdS}^{1+1}	Minkowskian \mathbf{M}^{1+1}	Doubly hyperbolic De Sitter \mathbf{dS}^{1+1}

On any general two-dimensional Riemannian V^2 or Lorentzian space L^2 , not necessarily of

constant curvature, there are two distinguished types of local coordinate systems, ‘geodesic parallel’ and ‘geodesic polar’, that reduce to the familiar Cartesian and polar coordinates on the Euclidean or Minkowskian plane [14, 37] (see Appendix). The κ -dependent Kepler problem in V_κ^2 was studied in [12] only in polar coordinates but, since the Euclidean oscillator allows separation also in parallel coordinates and this property is shared for its ‘curved’ version, we will use in this paper both types of ‘geodesic’ coordinates: polar (r, ϕ) and parallel (u, y) .

2.1 Polar coordinates

The following expression, where κ_1, κ_2 are two real parameters,

$$ds^2 = dr^2 + \kappa_2 S_{\kappa_1}^2(r) d\phi^2, \quad (2)$$

represents, in polar coordinates (r, ϕ) , the differential line element on the space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$. When $\kappa_2 = 1$, these spaces are, according to $\kappa_1 >, =, < 0$, the three *classical* Riemannian spaces $V_{\kappa_1}^2 \equiv \mathcal{S}_{\kappa_1}^2, \mathbf{E}^2, \mathbf{H}_{\kappa_1}^2$ with constant curvature κ_1 . In the three standard cases $\kappa_1 = 1, 0, -1$ the metrics correspond respectively to the *standard* sphere \mathbf{S}^2 ($\kappa_1 = 1, \kappa_2 = 1$), Euclidean plane \mathbf{E}^2 ($\kappa_1 = 0, \kappa_2 = 1$), and hyperbolic or Lobachevsky plane \mathbf{H}^2 ($\kappa_1 = -1, \kappa_2 = 1$):

$$ds^2|_{\mathbf{S}^2} = dr^2 + (\sin^2 r) d\phi^2, \quad ds^2|_{\mathbf{E}^2} = dr^2 + r^2 d\phi^2, \quad ds^2|_{\mathbf{H}^2} = dr^2 + (\sinh^2 r) d\phi^2,$$

Likewise, when $\kappa_2 = -1$ these spaces are the pseudo-Riemannian 2d spaces $L_{\kappa_1}^2$ with indefinite non-degenerate metric (hence Lorentzian) of constant curvature κ_1 and for the standard values $\kappa_1 = 1, 0, -1$ the metrics reduces to:

$$ds^2|_{\mathbf{AdS}^{1+1}} = dr^2 - (\sin^2 r) d\phi^2, \quad ds^2|_{\mathbf{M}^{1+1}} = dr^2 - r^2 d\phi^2, \quad ds^2|_{\mathbf{dS}^{1+1}} = dr^2 - (\sinh^2 r) d\phi^2,$$

which correspond to the three *standard* Lorentzian spaces $\mathbf{AdS}^{1+1}, \mathbf{M}^{1+1}, \mathbf{dS}^{1+1}$. Here polar coordinates only cover the region with ‘time-like’ separation from the origin; unlike the Riemannian case, \mathbf{AdS}^{1+1} and \mathbf{dS}^{1+1} are related by a change of sign in the metric, and are thus essentially the same space; this transformation is conveyed by the change $r \leftrightarrow ir$, which interchanges the regions with time-like and space-like separation from the origin.

The three vector fields X_{P_1}, X_{P_2}, X_J , whose coordinate expressions are given by:

$$\begin{aligned} X_{P_1} &= C_{\kappa_2}(\phi) \frac{\partial}{\partial r} - \frac{S_{\kappa_2}(\phi)}{T_{\kappa_1}(r)} \frac{\partial}{\partial \phi}, \\ X_{P_2} &= \kappa_2 S_{\kappa_2}(\phi) \frac{\partial}{\partial r} + \frac{C_{\kappa_2}(\phi)}{T_{\kappa_1}(r)} \frac{\partial}{\partial \phi}, \\ X_J &= \frac{\partial}{\partial \phi}, \end{aligned} \quad (3)$$

are Killing vector fields of the metric, and are well defined for any value of κ_1, κ_2 (even in the most degenerate CK space, the Galilean or isotropic space where $\kappa_1 = 0, \kappa_2 = 0$). Each X generates a one-parameter group of isometries of the metric and altogether close on a Lie algebra denoted $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$:

$$[X_J, X_{P_1}] = -X_{P_2} \quad [X_J, X_{P_2}] = \kappa_2 X_{P_1} \quad [X_{P_1}, X_{P_2}] = -\kappa_1 X_J. \quad (4)$$

Of course, when κ_1, κ_2 are set to any particular values, all these expressions give the pertinent ones for the corresponding spaces; this is the trait in all the CK formalism. Notice that only when $\kappa_1 = 0$ (Euclidean, Galilean and Minkowskian plane) X_{P_1} and X_{P_2} commute. If we restrict to the family of classical homogeneous Riemannian spaces $V_{\kappa_1}^2$ with curvature κ_1 , the corresponding Killing

vector fields are given by setting $\kappa_2=1$ in (3), and then the angular coordinate appears through the circular trigonometric functions in the three spaces, where the radial coordinate appears through the κ_1 -ones, which are either circular, parabolic or hyperbolic according to the sign of κ_1 :

$$\begin{aligned} X_{P_1}|_{V_\kappa^2} &= (\cos \phi) \frac{\partial}{\partial r} - \left(\frac{C_{\kappa_1}(r)}{S_{\kappa_1}(r)} \sin \phi \right) \frac{\partial}{\partial \phi}, \\ X_{P_1}|_{V_\kappa^2} &= (\sin \phi) \frac{\partial}{\partial r} + \left(\frac{C_{\kappa_1}(r)}{S_{\kappa_1}(r)} \cos \phi \right) \frac{\partial}{\partial \phi}, \\ X_J|_{V_\kappa^2} &= \frac{\partial}{\partial \phi}. \end{aligned}$$

Moreover, the Lagrangian for a (free) particle moving in a configuration space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ is given by the kinetic term invariant under the actions of X_{P_1} , X_{P_2} , X_J arising from the metric:

$$\mathcal{L}_0(r, \phi, v_r, v_\phi) = T_{(\kappa_1, \kappa_2)}(r, \phi, v_r, v_\phi) = \frac{1}{2} (v_r^2 + \kappa_2 S_{\kappa_1}^2(r) v_\phi^2),$$

A general *natural* Lagrangian in $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ (kinetic minus potential term) has the following form

$$\mathcal{L}(r, \phi, v_r, v_\phi) = \frac{1}{2} (v_r^2 + \kappa_2 S_{\kappa_1}^2(r) v_\phi^2) - \mathcal{V}(r, \phi),$$

in such a way that for $\kappa_1=0, \kappa_2=1$ we recover a standard Euclidean system

$$L(r, \phi, v_r, v_\phi) = \lim_{\kappa_1 \rightarrow 0, \kappa_2=1} \mathcal{L}(r, \phi, v_r, v_\phi) = \frac{1}{2} (v_r^2 + r^2 v_\phi^2) - V(r, \phi), \quad V(r, \phi) = \lim_{\kappa_1 \rightarrow 0, \kappa_2=1} \mathcal{V}(r, \phi).$$

In some particular cases a Lagrangian system can possess the Killing vector fields X_{P_1} , X_{P_2} , or X_J (or any linear combination of them) as exact Noether symmetries. If we denote by X^t the natural tangent lift to the tangent bundle (velocity phase space) of the vector field X and by θ_L the Cartan semibasic one-form [41]

$$\theta_L = \frac{\partial L}{\partial v_r} dr + \frac{\partial L}{\partial v_\phi} d\phi = v_r dr + \kappa_2 S_{\kappa_1}^2(r) v_\phi d\phi,$$

then the *basic* cases with exact Noether symmetries are the following:

1. If the potential $\mathcal{V}(r, \phi)$ is invariant under X_{P_1} , then $\mathcal{V}(r, \phi)$ should depend on (r, ϕ) only through an arbitrary function of the single variable $z_2 \equiv S_{\kappa_1}(r) S_{\kappa_2}(\phi)$ and then

$$P_1 = i(X_{P_1}^t) \theta_L = (C_{\kappa_2}(\phi)) v_r - \kappa_2 (C_{\kappa_1}(r) S_{\kappa_1}(r) S_{\kappa_2}(\phi)) v_\phi$$

is a constant of motion.

2. If the potential $\mathcal{V}(r, \phi)$ is invariant under X_{P_2} , then $\mathcal{V}(r, \phi)$ should be an arbitrary function of the single variable $z_1 \equiv S_{\kappa_1}(r) C_{\kappa_2}(\phi)$ only, and then

$$P_2 = i(X_{P_2}^t) \theta_L = \kappa_2 (S_{\kappa_2}(\phi)) v_r + \kappa_2 (C_{\kappa_1}(r) S_{\kappa_1}^2(r) C_{\kappa_2}(\phi)) v_\phi$$

is a constant of motion.

3. If the potential $\mathcal{V}(r, \phi)$ is invariant under X_J , then $\mathcal{V}(r, \phi)$ should be an arbitrary function of the single variable r only (\mathcal{V} is a central potential) and the constant of motion is:

$$J = i(X_J^t) \theta_L = \kappa_2 S_{\kappa_1}^2(r) v_\phi.$$

Several remarks are pertinent. First, the radial dependence in the momenta P_1, P_2, J appears through κ_1 -trigonometric functions, and hence is sensitive to the curvature; the angular dependence is carried through κ_2 -trigonometric functions and in the three *classical* ($\kappa_2=1$) Riemannian spaces it appears through $\cos \phi$ or $\sin \phi$, irrespectively of the curvature. Second, the quantities P_1, P_2, J , could be considered as the ordinary linear momenta and angular momentum of a particle moving in the configuration space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$. In terms of these, the kinetic energy $T_{(\kappa_1, \kappa_2)}$ can be rewritten as follows (it is the Casimir of the isometry algebra, [22])

$$T_{(\kappa_1, \kappa_2)} = \frac{1}{2} \frac{\kappa_2 P_1^2 + P_2^2 + \kappa_1 J^2}{\kappa_2},$$

showing that, on spaces of (constant) non-zero curvature κ_1 , the angular momentum has a contribution to the kinetic energy of the system, proportional to the curvature κ_1 . And third, the new quantities P_2 and J vanish *identically* when $\kappa_2=0$. This is linked to the singular character of the corresponding Lagrangian, as the metric is degenerate when $\kappa_2=0$. This singular case is however not *generically* singular, but only a very special limit of a regular system, and one may expect the geodesic motion to have precisely three non-trivial constants of motion linear in the velocities. When working in the general CK scheme, where we want to cover all CK spaces $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ (even when $\kappa_2=0$) this suggests to consider, instead of P_1, P_2, J , the quantities defined as

$$\mathcal{P}_1 := P_1, \quad \mathcal{P}_2 := \frac{P_2}{\kappa_2}, \quad \mathcal{J} := \frac{J}{\kappa_2}, \quad (5)$$

which will be called the *CK Noether momenta*; when $\kappa_2 \neq 0$ these are essentially equivalent to P_1, P_2, J but $\mathcal{P}_1, \mathcal{P}_2, \mathcal{J}$ are to be preferred because they remain non vanishing even in the limit $\kappa_2 \rightarrow 0$. These will be the momenta used in the rest of the paper; we remark that in the *classical* Riemannian spaces of constant curvature κ_1 the three Noether momenta *coincide* with P_1, P_2, J . In $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ the CK Noether momenta are:

$$\mathcal{P}_1 = C_{\kappa_2}(\phi) v_r - \kappa_2 C_{\kappa_1}(r) S_{\kappa_1}(r) S_{\kappa_2}(\phi) v_\phi, \quad (6)$$

$$\mathcal{P}_2 = S_{\kappa_2}(\phi) v_r + C_{\kappa_1}(r) S_{\kappa_1}(r) C_{\kappa_2}(\phi) v_\phi, \quad (7)$$

$$\mathcal{J} = S_{\kappa_1}^2(r) v_\phi. \quad (8)$$

In terms of these Noether momenta, the kinetic energy is well defined for all CK spaces, contains always a term \mathcal{P}_1^2 and is given by:

$$T_{(\kappa_1, \kappa_2)} = \frac{1}{2} (\mathcal{P}_1^2 + \kappa_2 \mathcal{P}_2^2 + \kappa_1 \kappa_2 \mathcal{J}^2). \quad (9)$$

2.2 Parallel coordinates

The element of arc length ds^2 in the space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ is given in parallel coordinates (u, y) by

$$ds^2 = C_{\kappa_1 \kappa_2}^2(y) du^2 + \kappa_2 dy^2, \quad (10)$$

and in the three particular classical Riemannian standard cases it reduces to

$$ds^2|_{\mathbf{S}^2} = (\cos^2 y) du^2 + dy^2, \quad ds^2|_{\mathbf{E}^2} = du^2 + dy^2, \quad ds^2|_{\mathbf{H}^2} = (\cosh^2 y) du^2 + dy^2.$$

The Lagrangian of a free particle in $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ has a kinetic term corresponding to the metric:

$$\mathcal{L}(u, y; v_u, v_y) = T_{(\kappa_1, \kappa_2)}(u, y; v_u, v_y) = \frac{1}{2} (C_{\kappa_1 \kappa_2}^2(y) v_u^2 + \kappa_2 v_y^2).$$

In these coordinates the three Killing vector fields closing the Lie algebra $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ are

$$\begin{aligned} X_{P_1} &= \frac{\partial}{\partial u}, \\ X_{P_2} &= \kappa_1 \kappa_2 S_{\kappa_1}(u) T_{\kappa_1 \kappa_2}(y) \frac{\partial}{\partial u} + C_{\kappa_1}(u) \frac{\partial}{\partial y}, \\ X_J &= -\kappa_2 C_{\kappa_1}(u) T_{\kappa_1 \kappa_2}(y) \frac{\partial}{\partial u} + S_{\kappa_1}(u) \frac{\partial}{\partial y}, \end{aligned} \quad (11)$$

and the associated momenta

$$\begin{aligned} P_1 &= i(X_{P_1}^t) \theta_L = C_{\kappa_1 \kappa_2}^2(y) v_u, \\ P_2 &= i(X_{P_2}^t) \theta_L = \kappa_1 \kappa_2 S_{\kappa_1}(u) C_{\kappa_1 \kappa_2}(y) S_{\kappa_1 \kappa_2}(y) v_u + \kappa_2 C_{\kappa_1}(u) v_y, \\ J &= i(X_J^t) \theta_L = -\kappa_2 C_{\kappa_1}(u) C_{\kappa_1 \kappa_2}(y) S_{\kappa_1 \kappa_2}(y) v_u + \kappa_2 S_{\kappa_1}(u) v_y, \end{aligned}$$

lead to the CK Noether momenta:

$$\mathcal{P}_1 = C_{\kappa_1 \kappa_2}^2(y) v_u, \quad (12)$$

$$\mathcal{P}_2 = \kappa_1 S_{\kappa_1}(u) S_{\kappa_1 \kappa_2}(y) C_{\kappa_1 \kappa_2}(y) v_u + C_{\kappa_1}(u) v_y, \quad (13)$$

$$\mathcal{J} = -C_{\kappa_1}(u) S_{\kappa_1 \kappa_2}(y) C_{\kappa_1 \kappa_2}(y) v_u + S_{\kappa_1}(u) v_y. \quad (14)$$

Notice how the expressions for the CK momenta specialize in the Euclidean case:

$$\mathcal{P}_1|_{\mathbf{E}^2} = v_u \quad \mathcal{P}_2|_{\mathbf{E}^2} = v_y \quad \mathcal{J}|_{\mathbf{E}^2} = u v_y - y v_u.$$

The general CK expressions can be looked at as a two-parameter deformation of the Galilean ones $\kappa_1 = 0, \kappa_2 = 0$, governed by the two constants κ_1, κ_2 ; the Euclidean case is *not* the natural comparison standard in the deformation, as one of the constants is already non-vanishing for \mathbf{E}^2 .

A potential \mathcal{V} , now expressed as a function of (u, y) , turns out to be invariant under X_{P_1} if \mathcal{V} is an arbitrary function of the single variable y only; this result is simply the translation to parallel coordinates of the previous result in polar coordinates—as consequence of the relation $S_{\kappa_1 \kappa_2}(y) = S_{\kappa_1}(r) S_{\kappa_2}(\phi)$ —, so that the auxiliary variable z_2 turns out to be precisely $S_{\kappa_1 \kappa_2}(y)$. Similar results describe the general form, in coordinates (u, y) of a potential invariant under X_{P_2} or under X_J .

We close this section with a comment on the expressions obtained for the three vector fields, X_{P_1}, X_{P_2}, X_J (for $\kappa_2 = 1$ this point was discussed in [51]). According to the straightening-out theorem [1], a vector field X on a n -manifold V always admits a local coordinate system $\{x_1, \dots, x_n\}$ in an appropriate neighbourhood of a regular point $X(m) \neq 0, m \in M$, such that then it becomes $Y = \sum_k c_k(\partial/\partial x_k)$, with $c_{k_0} = 1, c_k = 0$ for $k \neq k_0$. We recall that in polar coordinates X_J is given by $X_J = \partial/\partial\phi$ and now we have obtained that in parallel (u, y) coordinates X_{P_1} takes the form $X_{P_1} = \partial/\partial u$. Similarly, in the complementary ‘orthogonal’ parallel system (x, v) , we obtain $X_{P_2} = \partial/\partial v$ (this parallel system is not used in this paper but is discussed in the Appendix). So, these three coordinate systems, (r, ϕ) , (u, y) , and (x, v) , are the three appropriated systems (via the straightening-out theorem) providing the ‘straight’ expressions of X_J, X_{P_1} , and X_{P_2} , respectively.

3 The harmonic oscillator on $\mathcal{S}_{\kappa_1[\kappa_2]}^2$

The following Lagrangian in the CK space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ with curvature κ_1 and signature type κ_2

$$\mathcal{L}(r, \phi, v_r, v_\phi) = \frac{1}{2} (v_r^2 + \kappa_2 S_{\kappa_1}^2(r) v_\phi^2) - \mathcal{V}_{HO}(r), \quad \mathcal{V}_{HO} = \frac{1}{2} \omega_0^2 T_{\kappa_1}^2(r), \quad (15)$$

represents the ‘harmonic oscillator’ in the space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ [50, 51]; the potential $\mathcal{V}_{HO}(r)$ is ‘central’ in the sense it depends on the radial coordinate only; this dependence involves the label κ_1 (but not κ_2) and reduces to

$$\mathcal{V}_{HO}|_{\kappa_1=1} = \frac{1}{2} \omega_0^2 \tan^2 r, \quad \mathcal{V}_{HO}|_{\kappa_1=0} = V = \frac{1}{2} \omega_0^2 r^2, \quad \mathcal{V}_{HO}|_{\kappa_1=-1} = \frac{1}{2} \omega_0^2 \tanh^2 r,$$

which are the three harmonic oscillator potentials in the three classical standard Riemannian spaces $\mathbf{S}^2, \mathbf{E}^2, \mathbf{H}^2$. The Euclidean function $V(r)$ appears in this formalism as making a separation between two different behaviours (see Figure 1). In the sphere this potential was considered by Liebmann [39] (1905 edition), and later on by Higgs [28] and Leemon [38].

Recall r denotes the distance to the origin point computed in the intrinsic metric on the CK space. Thus in the classical Riemannian case ($V_{\kappa_1}^2, \kappa_2 = 1$), the potential has a zero (minimum) value at the origin and starts growing quadratically with the distance to the origin point, as implied by the approximation $T_{\kappa_1}(r) \approx r$ around $r = 0$, which holds for all values of κ_1 . In the flat case the potential grows quadratically with any r and approaches an infinite value only when $r \rightarrow \infty$. When the curvature is non-zero, the behaviour differs in a way depending on the curvature sign. When κ_1 is positive the potential grows faster and tends to infinity at a finite value $r = \frac{\pi}{2\sqrt{\kappa_1}}$, this is, on the sphere ‘equator’ (with the origin taken as the pole); the harmonic oscillator on the sphere splits the configuration space into two halves by an infinite potential wall on the equator, so the spherical harmonic oscillator has two antipodal centres. In the negative curvature case, the potential grows slower than in the flat case, and as $r \rightarrow \infty$ approaches a plateau, with a (positive) finite height $\mathcal{V}_\infty := \omega_0^2/(-2\kappa_1)$.

The motion in this potential is *superintegrable* in all CK spaces $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ since, in addition to the angular momentum \mathcal{J} , this system is endowed with the following quadratic constants of the motion

$$\begin{aligned} I_{\mathcal{J}^2} &= \mathcal{J}^2, \\ I_{\mathcal{P}_1^2} &= \mathcal{P}_1^2 + \mathcal{W}_{11}(r, \phi), & \mathcal{W}_{11}(r, \phi) &= \omega_0^2 T_{\kappa_1}^2(r) C_{\kappa_2}^2(\phi), \\ I_{\mathcal{P}_2^2} &= \mathcal{P}_2^2 + \mathcal{W}_{22}(r, \phi), & \mathcal{W}_{22}(r, \phi) &= \omega_0^2 T_{\kappa_1}^2(r) S_{\kappa_2}^2(\phi), \\ I_{\mathcal{P}_1 \mathcal{P}_2} &= \mathcal{P}_1 \mathcal{P}_2 + \mathcal{W}_{12}(r, \phi) & \mathcal{W}_{12}(r, \phi) &= \omega_0^2 T_{\kappa_1}^2(r) C_{\kappa_2}(\phi) S_{\kappa_2}(\phi). \end{aligned} \quad (16)$$

Remark that $\mathcal{W}_{11}(r, \phi), \mathcal{W}_{22}(r, \phi), \mathcal{W}_{12}(r, \phi)$ are well defined for any CK space, and they do not vanish identically in none of them.

The ‘energy’ of the motion can be written as:

$$I_E = \frac{1}{2} (I_{\mathcal{P}_1^2} + \kappa_2 I_{\mathcal{P}_2^2} + \kappa_1 \kappa_2 I_{\mathcal{J}^2}) = \frac{1}{2} (\mathcal{P}_1^2 + \kappa_2 \mathcal{P}_2^2 + \kappa_1 \kappa_2 \mathcal{J}^2) + \frac{1}{2} \omega_0^2 T_{\kappa_1}^2(r), \quad (17)$$

reducing on \mathbf{E}^2 to the known Euclidean expression. In the general CK space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ the kinetic energy is no longer given by (one half of) the ‘norm’ of the ‘momentum vector’ $\mathcal{P}_1^2 + \kappa_2 \mathcal{P}_2^2$ but contains as an *extra contribution* the square of the angular momentum, proportional to the curvature κ_1 and thus disappearing in the flat case $\kappa_1 = 0$.

The four integrals of motion in (16) cannot be functionally independent; indeed they satisfy the relation:

$$I_{\mathcal{P}_1^2} I_{\mathcal{P}_2^2} - (I_{\mathcal{P}_1 \mathcal{P}_2})^2 = \omega_0^2 I_{\mathcal{J}^2}.$$

Taken altogether the constants $I_{\mathcal{P}_1^2}, I_{\mathcal{P}_2^2}, I_{\mathcal{P}_1 \mathcal{P}_2}$ are the components of a (symmetric) tensor under the ‘rotation subgroup’ $SO_{\kappa_2}(2)$ in any space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$

$$\begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & \mathcal{F}_{22} \end{pmatrix} = \begin{pmatrix} I_{\mathcal{P}_1^2} & I_{\mathcal{P}_1 \mathcal{P}_2} \\ I_{\mathcal{P}_1 \mathcal{P}_2} & I_{\mathcal{P}_2^2} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_1^2 + \mathcal{W}_{11}(q_1, q_2) & \mathcal{P}_1 \mathcal{P}_2 + \mathcal{W}_{12}(q_1, q_2) \\ \mathcal{P}_1 \mathcal{P}_2 + \mathcal{W}_{12}(q_1, q_2) & \mathcal{P}_2^2 + \mathcal{W}_{22}(q_1, q_2) \end{pmatrix} \quad (18)$$

Thus the essential property of the Euclidean harmonic oscillator, to have a tensor constant of motion (the so called Fradkin tensor [18, 29]), survives for the ‘curved’ harmonic oscillator in any CK space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$. This tensor, looked at as the general CK form of the Fradkin tensor, contains also a complete set of functionally independent constants of motion, because $\mathcal{J}^2 = I_{\mathcal{J}^2}$ is related to the determinant of the Fradkin tensor in a ‘universal’ way, with a relation explicitly independent of κ_1, κ_2 :

$$\det(F) = \omega_0^2 I_{\mathcal{J}^2} \quad (19)$$

Another possible choice for three functionally independent constants is the set $\{I_{\mathcal{P}_1^2}, I_{\mathcal{P}_2^2}, I_{\mathcal{J}^2}\}$.

3.1 The classification of orbits and the equivalent one-dimensional problem

It is very convenient to introduce in the study of central potentials $V(r)$ on the Euclidean space a one-dimensional effective potential V^{eff} , which governs the radial motion after elimination of the ignorable angular coordinate. This procedure allows us to obtain a classification of orbits before the integration of the equations [19]. Given a central potential \mathcal{V} on the CK spaces $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ one can proceed similarly and this leads to the one-dimensional ‘effective’ potential $\mathcal{V}^{\text{eff}}(r)$:

$$\mathcal{V}^{\text{eff}}(r) = \mathcal{V}(r) + \frac{\kappa_2 \mathcal{J}^2}{2 \mathcal{S}_{\kappa_1}^2(r)},$$

where the extra term plays the role of the ‘centrifugal barrier potential’, reducing as it should be to $\mathcal{J}^2/(2r^2)$ in the standard Euclidean case. Therefore we can classify the orbits and obtain some additional information for the harmonic oscillator motion in any CK space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ simply by analyzing the effective potential for the harmonic oscillator motion:

$$\mathcal{V}^{\text{eff}}(r) = \frac{1}{2} \omega_0^2 \mathcal{T}_{\kappa_1}^2(r) + \frac{\kappa_2 \mathcal{J}^2}{2 \mathcal{S}_{\kappa_1}^2(r)} = \frac{1}{2} \omega_0^2 \mathcal{T}_{\kappa_1}^2(r) + \frac{\kappa_2 \mathcal{J}^2}{2 \mathcal{T}_{\kappa_1}^2(r)} + \frac{1}{2} \kappa_1 \kappa_2 \mathcal{J}^2, \quad (20)$$

which in the three particular one-dimensional problems associated to the standard ($\kappa_1 = 1, 0, -1$; $\kappa_2 = 1$) sphere \mathbf{S}^2 , Euclidean plane \mathbf{E}^2 , and Lobachevsky plane \mathbf{H}^2 reduces respectively to

$$\begin{aligned} \mathcal{V}_{HO}^{\text{eff}}|_{\mathbf{S}^2} &= \frac{1}{2} \omega_0^2 \tan^2 r + \frac{J^2}{2 \sin^2 r} = \frac{1}{2} \omega_0^2 \tan^2 r + \frac{J^2}{2 \tan^2 r} + \frac{1}{2} J^2, \\ \mathcal{V}_{HO}^{\text{eff}}|_{\mathbf{E}^2} &= \frac{1}{2} \omega_0^2 r^2 + \frac{J^2}{2r^2}, \\ \mathcal{V}_{HO}^{\text{eff}}|_{\mathbf{H}^2} &= \frac{1}{2} \omega_0^2 \tanh^2 r + \frac{J^2}{2 \sinh^2 r} = \frac{1}{2} \omega_0^2 \tanh^2 r + \frac{J^2}{2 \tanh^2 r} - \frac{1}{2} J^2 \end{aligned}$$

The standard Euclidean case $\kappa_1 = 0, \kappa_2 = 1$ needs no comment. We discuss in some detail the Riemannian cases, allowing any nonzero (positive) values for the curvature $\kappa_1 \neq 0$ and signature type $\kappa_2 > 0$. The discussion is made within the generic case $\mathcal{J} \neq 0$, and we will assume $\mathcal{J} > 0$ by considering if necessary the reversed motion along the same geometric orbit; for $\mathcal{J} = 0$ the motion is actually one-dimensional. From now on we also omit the label HO in the potential when it is clear from the context.

(1) Spherical case: Analysis of the potential \mathcal{V}^{eff} for $\kappa_1 > 0, \kappa_2 > 0$.

This corresponds to motion in a sphere, and the natural range for the r coordinate is the interval $(0, \pi/(\sqrt{\kappa_1}))$, which is the span of r along half a geodesic (half a sphere’s large circle). Provided ω_0

is real as implicitly assumed (the constant in \mathcal{V}_{HO} is assumed to be positive) the effective potential is always positive, $\mathcal{V}^{\text{eff}}(r) > 0$, satisfies the following limits in the boundaries

$$\lim_{r \rightarrow 0} \mathcal{V}^{\text{eff}}(r) = +\infty, \quad \lim_{r \rightarrow \pi/(2\sqrt{\kappa_1})} \mathcal{V}^{\text{eff}}(r) = +\infty,$$

and it has a minimum, whose value we denote $E(\mathcal{J})$, at the point r_m given by $T_{\kappa_1}^2(r_m) = \sqrt{\kappa_2}\mathcal{J}/\omega_0$

$$E(\mathcal{J}) := \mathcal{V}^{\text{eff}}(r_m) = \sqrt{\kappa_2}\omega_0\mathcal{J} + \frac{1}{2}\kappa_1\kappa_2\mathcal{J}^2.$$

This equivalent potential represents therefore, an asymmetrical well on the ‘ r line’, with two barriers of infinite height at $r = 0$ and $r = \pi/(2\sqrt{\kappa_1})$, and one single minimum placed in between (Figure 2). Thus, for a fixed value of \mathcal{J} the situation is as follows: there is not any possible motion for energies $E < E(\mathcal{J})$, there is the (unique) motion $r(t) = r_m$ when $E = E(\mathcal{J})$, and for all energy values $E > E(\mathcal{J})$ the motion of the radial coordinate consists of non-linear one-dimensional oscillations between the two ‘radial’ turning points. On the sphere, the motion with a fixed value of \mathcal{J} and the minimum compatible energy $E = E(\mathcal{J})$ is a circular motion, on the circle centred in the origin with radius $r = r_m$. The trajectories with $E(\mathcal{J}) < E$ lie in a spherical annulus which always contains the circle $r = r_m$; this annulus grows to cover all the hemisphere when $E \rightarrow \infty$.

All these expressions depend on κ_1, κ_2 and reduce to the well known expressions for the Euclidean oscillator when $\kappa_1 = 0, \kappa_2 = 1$; notice that the full Euclidean plane appears as the limit $\kappa_1 \rightarrow 0$ of a single hemisphere, with the sphere’s ‘equator’ going to the Euclidean infinity.

(2) Hyperbolic case: Analysis of the potential \mathcal{V}^{eff} for $\kappa_1 < 0, \kappa_2 > 0$.

This is the hyperbolic Lobachevsky plane case, and here $r \in (0, \infty)$. Provided again ω_0 is real as implicitly assumed, the effective potential is always positive, $\mathcal{V}^{\text{eff}}(r) > 0$, and has a minimum at $r = r_m$ whenever $T_{\kappa_1}(r_m) = \sqrt{\kappa_2}\mathcal{J}/\omega_0$. It satisfies the following limits in the boundaries:

$$\lim_{r \rightarrow 0} \mathcal{V}^{\text{eff}}(r) = +\infty, \quad \lim_{r \rightarrow +\infty} \mathcal{V}^{\text{eff}}(r) = \frac{\omega_0^2}{-2\kappa_1},$$

thus introducing into the problem a new energy scale, to be denoted E_∞

$$E_\infty := \frac{\omega_0^2}{-2\kappa_1};$$

(remark this scale could be also defined in the sphere case, where its value will anyhow fall outside of the physically allowed range of energies $[E(\mathcal{J}), \infty]$). The value of the potential at the minimum is given again by $E(\mathcal{J}) := \mathcal{V}^{\text{eff}}(r_m)$, but here the energy scale E_∞ is placed above $E(\mathcal{J})$: $E(\mathcal{J}) < E_\infty$

When $\kappa_1 < 0$ the (absolute) values of the hyperbolic type tangent $T_{\kappa_1}(r)$ are bounded by $\frac{1}{\sqrt{-\kappa_1}}$; thus depending on the values of \mathcal{J} there are two possible generic situations, according to whether \mathcal{J} is smaller or larger than an *angular momentum scale*

$$\mathcal{J}_\infty := \frac{\omega_0}{\sqrt{\kappa_2(-\kappa_1)}}$$

1. If $\mathcal{J} < \mathcal{J}_\infty$, this is if $\sqrt{\kappa_2}\mathcal{J}/\omega_0 < \frac{1}{-\kappa_1}$ the function $\mathcal{V}^{\text{eff}}(r)$ has a real minimum at the r_m given by $T_{\kappa_1}(r_m) = \sqrt{\kappa_2}\mathcal{J}/\omega_0$, with value $\mathcal{V}^{\text{eff}}(r_m) = \sqrt{\kappa_2}\omega_0\mathcal{J} + \frac{1}{2}\kappa_1\kappa_2\mathcal{J}^2$. Motions of the radial coordinate with the given value of \mathcal{J} will only happen with energies $E \geq E(\mathcal{J})$. Within this range of energies there are two possibilities. For $E(\mathcal{J}) < E < E_\infty$ the radial coordinate is bounded, corresponding to a motion in the hyperbolic plane in an annular region bounded by two circles (with radius corresponding to the two turning points in the effective one-dimensional radial potential); this annulus reduces to a circle of radius $r = r_m$ for $E = E(\mathcal{J})$.

When $E \rightarrow E_\infty$ while $E < E_\infty$ the exterior turning point approaches to infinity, reaches it for $E = E_\infty$ and then the exterior turning point disappears. Therefore for $E_\infty < E$ the motion of the radial coordinate is unbounded, corresponding to motions in the hyperbolic plane which stays outside of a geodesic circle centred at the origin, whose radius corresponds to the still existing interior turning point.

2. If $\mathcal{J} \geq \mathcal{J}_\infty$, this is if $\sqrt{\kappa_2}\mathcal{J}/\omega_0 \geq \frac{1}{\sqrt{-\kappa_1}}$ the function $\mathcal{V}^{\text{eff}}(r)$ has not any minimum, and decreases monotonically from the potential barrier with infinite height at the origin $r = 0$ to the asymptotic plateau when $r \rightarrow \infty$; only energies satisfying $E \geq E_\infty = \omega_0^2/(-2\kappa_1)$ will be allowed here and all the trajectories will be unbounded (scattering) open curves.

These two possible behaviours and its separating case are represented in Figure 3.

- (3) Lorentzian case: Analysis of the potential \mathcal{V}^{eff} for $\kappa_2 < 0$.

A complete discussion of motion in a Lorentzian configuration space will not be done here. We simply mention some traits. First, the ‘centrifugal barrier’ comes explicitly with the ‘opposite’ sign to the Riemannian one. Second, if r is the distance to the origin, the square r^2 can have any sign in a Lorentzian space (with $r = 0$ on the isotropes through the origin), and thus formally r will be real and positive on the region with time-like separation from the origin, but pure imaginary in the region with space-like separation. As the harmonic oscillator potential depends on $T_{\kappa_1}^2(r)$, the potential will always be real, yet it has the two signs on the two regions with time-like and space-like separation from the origin. Hence the harmonic oscillator potential has no isolated minima at any proper point, and all points on the isotropes through the origin are extremal, degenerate saddle points for the potential function. The possible extremal values for the effective potential are still given by the equation $T_{\kappa_1}(r_m) = \sqrt{\kappa_2}\mathcal{J}/\omega_0$. Because of the inexistence of isolated minima for the potential, there is no reason to require a positive ω_0^2 . If we accept a negative sign for the constant $\frac{1}{2}\omega_0^2$, (i.e., purely imaginary values for ω_0 , the products $\omega_0\sqrt{\kappa_2}$ and the quotients $\sqrt{\kappa_2}/\omega_0$ would turn real. In spite of these seemingly strange properties, the Lorentzian harmonic oscillator is superintegrable, and thus completely explicit solutions for its motion can be explicitly given, irrespective of any possible physical meanings which will be very different from the familiar oscillator. We just remind that the ‘inverted’ or repulsive Euclidean harmonic oscillator (with $\omega_0^2 < 0$) is also superintegrable, yet it only has scattering motions along centred hyperbolas, which are also conics with a centre at the origin; hence the complete set of conics with centre at the origin appear as the complete set of orbits in the harmonic oscillator only if we allow general, unrestricted, values, for the strength constants.

3.2 Determination of the orbits of the harmonic oscillator in $S_{\kappa_1[\kappa_2]}^2$

3.2.1 Method I: Direct Integration for a central potential

We have previously obtained, making use of the conservation of the total energy E and of the angular momentum \mathcal{J} two expressions for \dot{r} and $\dot{\phi}$. Eliminating t between both equations we have

$$d\phi = \frac{J dr}{S_{\kappa_1}^2(r) \sqrt{R(r)}}, \quad R(r) = 2 \left(E - \mathcal{V}(r) - \frac{\kappa_2 \mathcal{J}^2}{2 S_{\kappa_1}^2(r)} \right).$$

After the change of variable $r \rightarrow v$ with $v = 1/T_{\kappa_1}(r)$, $dr = -S_{\kappa_1}^2(r) dv$, this becomes

$$d\phi = -\frac{dv}{\sqrt{\widehat{R}(v)}}, \quad \widehat{R}(v) = \frac{2E}{\mathcal{J}^2} - \frac{2\mathcal{V}(v)}{\mathcal{J}^2} - \kappa_2(v^2 + \kappa_1).$$

All this holds for a general \mathcal{V} . Particularizing for the harmonic oscillator in $\mathbf{S}_{\kappa_1[\kappa_2]}^2$, $\mathcal{V}(v) = \frac{1}{2}\omega_0^2/v^2$

$$\widehat{R}(v) = \frac{2E_P}{\mathcal{J}^2} - \frac{\omega_0^2}{\mathcal{J}^2} \frac{1}{v^2} - \kappa_2 v^2,$$

where we have used the notation $E_P = E - \frac{1}{2}\kappa_1\kappa_2\mathcal{J}^2$. An integration leads to

$$\phi - \phi_0 = - \int \frac{dv}{\sqrt{\widehat{R}(v)}},$$

and the change $\chi = v^2$

$$\phi - \phi_0 = - \frac{1}{2} \int \frac{d\chi}{\sqrt{\widehat{R}'(\chi)}}, \quad \widehat{R}'(\chi) = -\frac{\omega_0^2}{\mathcal{J}^2} + \frac{2E_P}{\mathcal{J}^2}\chi - \kappa_2\chi^2,$$

makes the integration elementary. In this way we arrive at the general solution of the form

$$\frac{1}{T_{\kappa_1}^2(r)} = D - G \, C_{\kappa_2}(2(\phi - \phi_0)), \quad (21)$$

where of course the constant ϕ_0 is trivial in the sense that orbits with different values for ϕ_0 are permuted by isometries of the space, as corresponds to the central nature of the potential. The two constants D, G should be related to the values of the energy and angular momentum E, \mathcal{J} as it follows from the integration:

$$D = \frac{E_P}{\kappa_2\mathcal{J}^2}, \quad G = \frac{1}{\kappa_2\mathcal{J}^2} \sqrt{E_P^2 - \kappa_2\omega_0^2\mathcal{J}^2}, \quad E_P = E - \frac{1}{2}\kappa_1\kappa_2\mathcal{J}^2. \quad (22)$$

Hence the particular orbit corresponding to energy E and angular momentum \mathcal{J} is:

$$\frac{1}{T_{\kappa_1}^2(r)} = \frac{E_P}{\kappa_2\mathcal{J}^2} - \frac{1}{\kappa_2\mathcal{J}^2} \sqrt{E_P^2 - \kappa_2\omega_0^2\mathcal{J}^2} \, C_{\kappa_2}(2(\phi - \phi_0)), \quad (23)$$

which can be also rewritten introducing two new constants A, B as:

$$\frac{1}{T_{\kappa_1}^2(r)} = \frac{C_{\kappa_2}^2(\phi - \phi_0)}{A^2} + \frac{S_{\kappa_2}^2(\phi - \phi_0)}{B^2}, \quad (24)$$

where

$$\frac{1}{A^2} = D - G \quad \frac{1}{B^2} = \kappa_2(D + G), \quad (25)$$

are related to energy and angular momentum by the relations (to be stated later in a simpler form)

$$\frac{1}{A^2} = \frac{E_P - \sqrt{E_P^2 - \kappa_2\omega_0^2\mathcal{J}^2}}{\kappa_2\mathcal{J}^2}, \quad \frac{1}{B^2} = \frac{E_P + \sqrt{E_P^2 - \kappa_2\omega_0^2\mathcal{J}^2}}{\kappa_2\mathcal{J}^2}. \quad (26)$$

Note that in the Riemannian case (when $\kappa_2 > 0$), then $0 < G < D$, and therefore both $D - G$ and $D + G$ are positive, thus making well adapted the notation we have chosen for A^2, B^2 ; in the Lorentzian case things behave differently.

All these equations apply for any CK space. In particular, in the three classical *standard* Riemannian spaces ($\mathbf{S}^2, \mathbf{E}^2, \mathbf{H}^2$) with curvature $\kappa_1 = 1, 0, -1$; $\kappa_2 = 1$, and choosing the origin for ϕ so that $\phi_0 = 0$, the orbit equations are:

$$\begin{aligned} \frac{1}{\tan^2 r} &= \frac{1}{A^2} \cos^2 \phi + \frac{1}{B^2} \sin^2 \phi & \text{in } \mathbf{S}^2, \\ \frac{1}{r^2} &= \frac{1}{A^2} \cos^2 \phi + \frac{1}{B^2} \sin^2 \phi & \text{in } \mathbf{E}^2, \\ \frac{1}{\tanh^2 r} &= \frac{1}{A^2} \cos^2 \phi + \frac{1}{B^2} \sin^2 \phi & \text{in } \mathbf{H}^2, \end{aligned}$$

Let us close this direct integration approach with three observations. First, it turns out that in any of the nine CK spaces, the HO orbit is always a *conic* with centre at the potential origin, where ‘conic’ has to be taken in a metric sense, relative to the intrinsic metric in each space. This follows from the geometrical study to be presented in Section 4. In the Euclidean case the quantities A and B are directly the ellipse semiaxes; in the case $\kappa_1 \neq 0$ the semiaxes (understood in terms of the intrinsic metric) are related to A, B by some relations which will involve κ_1, κ_2 . Second, the connection between the coefficients D, G or A, B and the energy and momentum, when reexpressed in terms of the quantity E_P does not depend on κ_1 . Hence in the sphere or in the hyperbolic plane this relation has the same form as in Euclidean space; something similar happens in the Kepler problem in Riemannian curved spaces [12]. Third, both the method and the results obtained show a close similarity with the Euclidean ones. The classical and well known change of variable $r \rightarrow v = 1/r$ admits as a generalization the κ_1 -dependent change $r \rightarrow v = 1/T_{\kappa_1}(r)$ which affords a significant simplification for all values of κ_1 ; of course this change reduces to the Euclidean one $r \rightarrow v = 1/r$ for $\kappa_1 = 0$.

3.2.2 Method II: Equation of Binet

The expression (8) of the angular momentum \mathcal{J} determines a relation between the differentials of the time t and the angle ϕ

$$\mathcal{J} dt = S_{\kappa_1}^2(r) d\phi.$$

The corresponding relation between the derivatives with respect to t and ϕ is

$$\frac{d}{dt} = \left(\frac{\mathcal{J}}{S_{\kappa_1}^2(r)} \right) \frac{d}{d\phi},$$

so that the second derivative with respect to t is given by

$$\frac{d^2}{dt^2} = \left(\frac{\mathcal{J}}{S_{\kappa_1}^2(r)} \right) \frac{d}{d\phi} \left[\left(\frac{\mathcal{J}}{S_{\kappa_1}^2(r)} \right) \frac{d}{d\phi} \right].$$

Introducing this in the radial equation $\ddot{r} = \kappa_2 S_{\kappa_1}(r) C_{\kappa_1}(r) \dot{\phi}^2 - \frac{d\mathcal{V}}{dr}$, it becomes

$$\left(\frac{\mathcal{J}}{S_{\kappa_1}^2(r)} \right) \frac{d}{d\phi} \left[\left(\frac{\mathcal{J}}{S_{\kappa_1}^2(r)} \right) \frac{dr}{d\phi} \right] = \frac{\mathcal{J}^2}{T_{\kappa_1}(r)} \left(\frac{1}{S_{\kappa_1}^2(r)} \right) - \frac{d\mathcal{V}}{dr}.$$

This equation can be simplified in two steps: the term in brackets in the l.h.s. can be rewritten by making use of

$$\left(\frac{\mathcal{J}}{S_{\kappa_1}^2(r)} \right) \frac{dr}{d\phi} = -\mathcal{J} \frac{d}{d\phi} \left(\frac{1}{T_{\kappa_1}(r)} \right)$$

to obtain

$$-\mathcal{J}^2 \frac{d^2}{d\phi^2} \left(\frac{1}{T_{\kappa_1}(r)} \right) = \kappa_2 \mathcal{J}^2 \frac{1}{T_{\kappa_1}(r)} + \frac{d\mathcal{V}}{dr},$$

and then we introduce the change $r \rightarrow v$ with the potential $\mathcal{V}(v)$ considered as a function of v . In this way we arrive at the differential equation of the orbit

$$\frac{d^2 v}{d\phi^2} + \kappa_2 v = -\frac{1}{\mathcal{J}^2} \frac{d\mathcal{V}}{dv}$$

that permits us to obtain ϕ as a function of v for the given potential considered as function of v :

$$\phi - \phi_0 = \pm \int \left\{ c - \left(\frac{2}{\mathcal{J}^2} \right) \mathcal{V} - \kappa_2 v^2 \right\}^{-(1/2)} dv.$$

(A \pm sign is pertinent if both signs of the angular momentum along a given orbit are considered; notice this equation coincides with the one obtained in the previous subsection) Let us now particularize for the harmonic oscillator $\mathcal{V} = \frac{1}{2} \omega_0^2 (1/v^2)$ in the CK space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$: the equation itself reduces to a nonlinear equation of Pinney–Ermakov type

$$\frac{d^2 v}{d\phi^2} + \kappa_2 v = \frac{\omega_0^2}{\mathcal{J}^2} \frac{1}{v^3}$$

whose general solution, further to the parameter \mathcal{J} already present in the equation depends on two independent integration constants D, ϕ_0 and has the form

$$v = \sqrt{D - G C_{\kappa_2}(2(\phi - \phi_0))}, \quad G = \sqrt{D^2 - \frac{\omega_0^2}{\kappa_2 \mathcal{J}^2}} \quad (27)$$

which coincides with the general orbit obtained before. The differential equation of the orbit for the variable v , usually known in the Euclidean case as *Binet's Equation*, is essentially preserved by the κ_1, κ_2 -deformation. Indeed, for the three classical Riemannian spaces ($\kappa_2=1$) the new variable $v = 1/r$ deforms to $v = 1/T_{\kappa_1}(r)$ but the equation by itself remains invariant.

3.2.3 Method III: Superintegrability in parallel coordinates

In term of the parallel coordinates (u, y) the Lagrangian which represents the motion of a particle under an harmonic oscillator potential \mathcal{V} in the CK space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ is:

$$\mathcal{L}(u, y, v_u, v_y) = \frac{1}{2} (C_{\kappa_1 \kappa_2}^2(y) v_u^2 + \kappa_2 v_y^2) - \frac{1}{2} \omega_0^2 \mathcal{V}(u, y), \quad \mathcal{V}(u, y) = \frac{T_{\kappa_1}^2(u)}{C_{\kappa_1 \kappa_2}^2(y)} + \kappa_2 T_{\kappa_1 \kappa_2}^2(y) \quad (28)$$

The expression of the potential, which correspond to the function on $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ given previously in polar coordinates, displays its separability in (u, y) coordinates. The potential reduces to

$$\mathcal{V}|_{\mathbf{S}^2} = \frac{1}{2} \omega_0^2 \left(\frac{\tan^2 u}{\cos^2 y} + \tan^2 y \right) \quad \mathcal{V}|_{\mathbf{H}^2} = \frac{1}{2} \omega_0^2 \left(\frac{\tanh^2 u}{\cosh^2 y} + \tanh^2 y \right)$$

in the two particular cases of the *standard* unit sphere \mathbf{S}^2 and Lobachevsky plane \mathbf{H}^2 , and to

$$\mathcal{V}|_{\mathbf{E}^2} = V = \frac{1}{2} \omega_0^2 (u^2 + y^2) \equiv \frac{1}{2} \omega_0^2 (x^2 + y^2)$$

in the Euclidean case (where we recall the equality $u \equiv x$ for $\kappa_1=0$).

Since we have already solved the dynamics in polar coordinates (r, ϕ) , we can make use of the expressions relating parallel with polar coordinates (where the positive ‘ u axis’ $y = 0$ is taken to coincide with the polar axis $\phi = 0$; see Appendix). Then the orbit equation (taking $\phi_0 = 0$ to avoid inessential complications)

$$\left(E_P - \sqrt{E_P^2 - \kappa_2 \omega_0^2 \mathcal{J}^2} \right) T_{\kappa_1}^2(r) C_{\kappa_2}^2(\phi) + \left(E_P + \sqrt{E_P^2 - \kappa_2 \omega_0^2 \mathcal{J}^2} \right) T_{\kappa_1}^2(r) C_{\kappa_2}^2(\phi) = \kappa_2 \mathcal{J}^2,$$

becomes when written in coordinates (u, y)

$$\left(E_P - \sqrt{E_P^2 - \kappa_2 \omega_0^2 J^2} \right) T_{\kappa_1}^2(u) + \left(E_P + \sqrt{E_P^2 - \kappa_2 \omega_0^2 J^2} \right) \left(\frac{T_{\kappa_1 \kappa_2}(y)}{C_{\kappa_1}(u)} \right)^2 = \kappa_2 \mathcal{J}^2,$$

or

$$\frac{1}{A^2} T_{\kappa_1}^2(u) + \frac{1}{B^2} \frac{T_{\kappa_1 \kappa_2}^2(y)}{C_{\kappa_1}^2(u)} = 1. \quad (29)$$

The coefficients A, B are related to the values of the constants of motion. By using the identity

$$\frac{T_{\kappa_1}^2(u)}{C_{\kappa_1\kappa_2}^2(y)} + \kappa_2 T_{\kappa_1\kappa_2}^2(y) = T_{\kappa_1}^2(u) + \kappa_2 \frac{T_{\kappa_1\kappa_2}^2(y)}{C_{\kappa_1}^2(u)},$$

the energy constant

$$E = \frac{1}{2}(\mathcal{P}_1^2 + \kappa_2 \mathcal{P}_2^2 + \kappa_1 \kappa_2 \mathcal{J}^2) + \frac{1}{2}\omega_0^2 \left(\frac{T_{\kappa_1}^2(u)}{C_{\kappa_1\kappa_2}^2(y)} + \kappa_2 T_{\kappa_1\kappa_2}^2(y) \right), \quad (30)$$

can be rewritten as a linear combination of the constants related to the superintegrability of the harmonic oscillator:

$$I_E = \frac{1}{2}(I_{\mathcal{P}_1^2} + \kappa_2 I_{\mathcal{P}_2^2} + \kappa_1 \kappa_2 I_{\mathcal{J}^2})$$

where the quadratic constants of the motion specific to the harmonic oscillator in the space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ are:

$$\begin{aligned} I_{\mathcal{J}^2} &= \mathcal{J}^2 \\ I_{\mathcal{P}_1^2} &= \mathcal{P}_1^2 + \mathcal{W}_{11}(r, \phi), & \mathcal{W}_{11}(u, y) &= \omega_0^2 T_{\kappa_1}^2(u) \\ I_{\mathcal{P}_2^2} &= \mathcal{P}_2^2 + \mathcal{W}_{22}(r, \phi) & \mathcal{W}_{22}(u, y) &= \omega_0^2 \frac{T_{\kappa_1\kappa_2}^2(y)}{C_{\kappa_1}^2(u)} \\ I_{\mathcal{P}_1\mathcal{P}_2} &= \mathcal{P}_1\mathcal{P}_2 + \mathcal{W}_{12}(u, y) & \mathcal{W}_{12}(r, \phi) &= \omega_0^2 \frac{T_{\kappa_1}(u) T_{\kappa_1\kappa_2}(y)}{C_{\kappa_1}(u)}. \end{aligned} \quad (31)$$

As in polar coordinates, all the functions $\mathcal{W}_{11}(u, y), \mathcal{W}_{22}(u, y), \mathcal{W}_{12}(u, y)$ are defined for any CK space, and they do not vanish identically in none of them. In the Euclidean case, the CK momenta $\mathcal{P}_1, \mathcal{P}_2$ and the constants of motion reduce as they should to:

$$\mathcal{P}_1|_{\mathbf{E}^2} = v_u \quad \mathcal{P}_2|_{\mathbf{E}^2} = v_y \quad I_{\mathcal{P}_1^2}|_{\mathbf{E}^2} = v_u^2 + \omega_0^2 u^2, \quad I_{\mathcal{P}_1\mathcal{P}_2}|_{\mathbf{E}^2} = v_u v_y + \omega_0^2 uy, \quad I_{\mathcal{P}_2^2}|_{\mathbf{E}^2} = v_y^2 + \omega_0^2 y^2. \quad (32)$$

where once more we recall that for $\kappa_1=0$, we have $u \equiv x$. These constants are the elements of the Fradkin tensor [18, 29].

Back to the CK general case, and defining:

$$2E_1 := I_{\mathcal{P}_1^2} = \mathcal{P}_1^2 + \omega_0^2 T_{\kappa_1}^2(u) \quad (33)$$

$$2E_2 := I_{\mathcal{P}_2^2} = \mathcal{P}_2^2 + \omega_0^2 \frac{T_{\kappa_1\kappa_2}^2(y)}{C_{\kappa_1}^2(u)} \quad (34)$$

the energy can be written as

$$E = E_1 + \kappa_2 E_2 + \frac{1}{2}\kappa_1 \kappa_2 \mathcal{J}^2$$

where E_1, E_2 are the curved analogues to the ‘partial’ energies associated to the one-dimensional harmonic motions whose linear superposition provides the most general 2d Euclidean harmonic oscillator motion; remind that a curved configuration space is not an affine space, thus strictly speaking there is no a well defined way to superpose motions. The value of the remaining constant of motion $I_{\mathcal{P}_1\mathcal{P}_2}$ is equal to zero for orbits with $\phi_0 = 0$ (this choice amounts to diagonalize the Fradkin tensor, and $2E_1, 2E_2$ are the eigenvalues). In this case, the relation (19) gives

$$4E_1 E_2 = \omega_0^2 \mathcal{J}^2,$$

henceforth implying

$$(E_1 + \kappa_2 E_2)^2 - (-E_1 + \kappa_2 E_2)^2 = 4\kappa_2 E_1 E_2 = \kappa_2 \omega_0^2 \mathcal{J}^2. \quad (35)$$

The quantity E_P introduced before reduces simply to the ‘sum’ of the two E_1, E_2 contributions

$$E_P = E_1 + \kappa_2 E_2$$

and some simple algebra leads to neater expressions for the relations among physical constants and conic coefficients:

$$D = \frac{E_1 + \kappa_2 E_2}{\kappa_2 \mathcal{J}^2}, \quad G = \frac{-E_1 + \kappa_2 E_2}{\kappa_2 \mathcal{J}^2} \quad A^2 = \frac{2E_1}{\omega_0^2} \quad B^2 = \frac{2E_2}{\omega_0^2}. \quad (36)$$

Direct observation in the expressions for the ‘superintegrability constants’ leads also to the following identity, holding for all $\mathcal{S}_{\kappa_1[\kappa_2]}^2$:

$$\mathcal{F}_{22} T_{\kappa_1}^2(u) - 2\mathcal{F}_{12} T_{\kappa_1}(u) \left(\frac{T_{\kappa_1 \kappa_2}(y)}{C_{\kappa_1}(u)} \right) + \mathcal{F}_{11} \left(\frac{T_{\kappa_1 \kappa_2}(y)}{C_{\kappa_1}(u)} \right)^2 = \kappa_2 \mathcal{J}^2. \quad (37)$$

This property is proved by direct computation, and can be interpreted as a relation which must hold between the coordinates (u, y) along a given motion, so this is precisely the orbit equation, which can thus be obtained *directly* from the superintegrable character. This is the most general orbit, but if the coordinate axes are chosen so that the Fradkin tensor is diagonal, then $I_{\mathcal{P}_1 \mathcal{P}_2} = 0$ and the equation coincides, of course, with (29). In the particular $\kappa_1 = 0, \kappa_2 = 1$ Euclidean case we obtain

$$\mathcal{F}_{22} x^2 - 2\mathcal{F}_{12} xy + \mathcal{F}_{11} y^2 = \mathcal{J}^2$$

that, as it is well known, represents an ellipse in the Euclidean plane \mathbf{E}^2 .

3.3 The period of the harmonic oscillator

The Euclidean oscillator is the classical example of an isochronous system, with the same period $T = 2\pi/\omega_0$ for all orbits. A natural question is whether or not the ‘curved’ oscillator also inherits this property. We discuss this problem for the closed orbits in Riemannian configuration spaces, where $\kappa_2 > 0$. The strategy consists in choosing the orbits with $\phi_0 = 0$, and computing the time spent by the particle between the points $\phi = 0$ and $\phi = \frac{\pi}{2\sqrt{\kappa_2}}$, which will always be reached along a closed orbit. By symmetry reasons this time equals one fourth of the orbit period.

The angular velocity, taken from the angular momentum first integral:

$$\dot{\phi} \equiv \frac{d\phi}{dt} = \frac{\mathcal{J}}{\mathcal{S}_{\kappa_1}^2(r)},$$

can be rewritten by successively using the identity

$$\frac{1}{\mathcal{S}_{\kappa_1}^2(r)} = \frac{1}{\mathcal{T}_{\kappa_1}^2(r)} + \kappa_1,$$

and the orbit equation

$$\frac{1}{\mathcal{T}_{\kappa_1}^2(r)} = \frac{\omega_0^2}{2E_1} C_{\kappa_2}^2(\phi) + \frac{\omega_0^2}{2E_2} \mathcal{S}_{\kappa_2}^2(\phi),$$

as

$$\frac{1}{\mathcal{J}} \frac{d\phi}{dt} = \frac{\omega_0^2}{2E_1} C_{\kappa_2}^2(\phi) + \frac{\omega_0^2}{2E_2} \mathcal{S}_{\kappa_2}^2(\phi) + \kappa_1,$$

where by introducing double angles and simplifying we get:

$$\frac{4\kappa_2 E_1 E_2}{\mathcal{J} \omega_0^2} \frac{d\phi}{dt} = \kappa_2 \mathcal{J} \frac{d\phi}{dt} = \left(E_1 + \kappa_2 E_2 + \kappa_1 \kappa_2 \mathcal{J}^2 \right) + \left(-E_1 + \kappa_2 E_2 \right) C_{\kappa_2}(2\phi).$$

By integrating along one fourth of a complete closed orbit:

$$T = 4 \int_0^{\frac{\pi}{2\sqrt{\kappa_2}}} dt = 4\kappa_2 \mathcal{J} \int_0^{\frac{\pi}{2\sqrt{\kappa_2}}} \frac{d\phi}{\alpha + \beta \cos(2\sqrt{\kappa_2}\phi)},$$

where α and β denote the following expressions:

$$\alpha := (E_1 + \kappa_2 E_2 + \kappa_1 \kappa_2 \mathcal{J}^2) \quad \beta := (-E_1 + \kappa_2 E_2).$$

The change $\zeta = 2\sqrt{\kappa_2}\phi$ makes the integration elementary

$$T = 2\sqrt{\kappa_2} \mathcal{J} \int_0^\pi \frac{d\zeta}{\alpha + \beta \cos \zeta} = 2\sqrt{\kappa_2} \mathcal{J} \frac{\pi}{\sqrt{\alpha^2 - \beta^2}}.$$

Now some algebra and the use of the relations (35) leads to

$$T = \frac{2\pi}{\omega_0} \frac{1}{\sqrt{1 + \frac{2\kappa_1 E}{\omega_0^2}}},$$

an *exact* result whose expansion in powers of the curvature has the correct Euclidean period as the zero-th order term, with a fractional first-order correction by the dimensionless quotient $\kappa_1 E/\omega_0^2$:

$$T = \frac{2\pi}{\omega_0} \left(1 - \frac{E}{\omega_0^2} \kappa_1 + \dots\right), \quad T|_{\kappa_1=0} = \frac{2\pi}{\omega_0}.$$

Thus when the configuration space has curvature the period ceases to be the same for all orbits (as it was to be expected), yet it depends *only* on the total energy; this property is characteristic for all closed orbits of superintegrable systems. On the sphere, the period tends to infinity only when the orbit approaches the equator, where $E \rightarrow \infty$, but on the hyperbolic plane one might expect the period to diverge precisely when the orbit character changes from a closed orbit to scattering open orbit. From the previous section we know this happens for $E = E_\infty$ and this is precisely the value which makes $1 + \frac{2\kappa_1 E}{\omega_0^2} = 0$, and hence $T = \infty$. For energies above E_∞ the derivation as provided is not strictly applicable (because in these cases the orbit does not reach any point with $\phi = \frac{\pi}{2\sqrt{\kappa_2}}$; this is reflected in the formally imaginary result got for T in these cases).

4 Conics in spaces of constant curvature

In this section we give a brief geometric description of conics in the CK spaces, focusing on the three classical Riemannian ($\kappa_2 > 0$) constant curvature spaces $\mathbf{S}_{\kappa_1}^2, \mathbf{E}^2, \mathbf{H}_{\kappa_1}^2$ and emphasizing only those aspects relevant in relation with oscillator motion in these spaces. This description is intended to be self-contained, but as far as conics in these spaces are concerned, it is also complementary to comments made in our previous paper devoted to Kepler problem [12].

We start by recalling the basics. Any $\mathbf{S}_{\kappa_1[\kappa_2]}^2$ is a homogeneous symmetric space, and any homogeneous symmetric space has a canonical connection which is always unique and well defined. Restricting to the homogeneous spaces $\mathbf{S}_{\kappa_1[\kappa_2]}^2$, and due to the quasi-simplicity of the involved Lie algebras $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$, there is always an invariant metric \mathbf{g} in $\mathbf{S}_{\kappa_1[\kappa_2]}^2$ which is non-degenerate whenever $\kappa_2 \neq 0$. The canonical connection is always compatible with the metric. Generically, i.e., for $\kappa_2 \neq 0$, we have more: the canonical connection of $\mathbf{S}_{\kappa_1[\kappa_2]}^2$ as symmetric space coincides with the Levi-Civita (metric) connection associated to the metric. In the degenerate case $\kappa_2 = 0$ however, there are many connections compatible with the (degenerate) metric, and the canonical connection of the homogeneous symmetric space is singled out among the many connections compatible with the metric.

By *lines* will mean the *autoparallels* of this canonical connection. Generically (when $\kappa_2 \neq 0$), these coincide with the extremal curves of the length functional. In the Lorentzian case there are two generic types of geodesics: time-like, with real length and space-like, with ‘imaginary’ length relative to the main metric \mathbf{g} (alternatively one may think of another space-like length along space-like curves, corresponding to the ‘companion metric’ \mathbf{g}/κ_2 ; the space-like curves have a real ‘length’ as computed relative to this metric yet this choice will not be used in this paper). Thus from now on, all distances between points or between a point and a line will refer to lengths along geodesics measured relatively to the main metric (hence when $\kappa_2 < 0$ distances might be either real, vanishing or pure imaginary).

The *geometric* definition of conics, which makes sense in any 2-d space of constant curvature κ_1 and non-degenerate metric ($\kappa_2 \neq 0$) involves *focal elements*, i.e., either points which are assumed oriented, or lines which can be oriented and co-oriented. In any such space, and by definition:

An *ellipse/hyperbola* will be the set of points with a constant sum/difference $2a$ of distances r_1, r_2 to two fixed points F_1, F_2 separated a distance $2f$ and called *foci*.

A *parabola* will be the set of points with a constant sum/difference of distances r_1, \tilde{r}_2 to a fixed point F_1 , called *focus*, and to a fixed line f_2 , called *focal line*, (\tilde{r}_2 is assumed to be oriented); the *oriented* distance between F_1 and f_2 plays here the role of focal separation.

An *ultraellipse/ultrahyperbola* will be the set of points with a constant sum/difference $2a$ of oriented distances \tilde{r}_1, \tilde{r}_2 to two fixed intersecting lines f_1, f_2 separated by an angle $2F$ and called *focal lines*.

In the generic CK 2-d space of constant curvature $\kappa_1 \neq 0, \kappa_2 \neq 0$ these three pairs of curves, each pair sharing the same focal elements, are the *generic* conics, and all the remaining possible conics are either particular instances with focal separation vanishing ($f = 0, \phi = 0, F = 0$) or limiting cases, where some conic elements go to infinity (if possible at all); both particular and limiting cases can be obtained as suitable limits from the generic conics.

An important observation is that the Euclidean plane is *not generic* neither among the complete family of CK spaces, nor among the restricted family of Riemannian constant curvature planes. Thus some Euclidean properties of conics are very special and do not provide a good viewpoint to assess the $\kappa_1 \neq 0$ properties. For instance, in the *hyperbolic plane* $\mathbf{H}_{\kappa_1}^2$ a given ultraellipse, determined according to its definition by a pair of intersecting focal lines, has not only this pair of focal lines, but altogether *three* such pairs, the remaining two pairs being the non-intersecting pairs of focal lines obtained by joining in the two possible ways the endpoints of the (intersecting) initial pair. The distances between the two members of each pair plays the role of focal separations in such cases; and are in a suitable sense complementary (precisely, the paralelism angles of half these distances are complementary in the ordinary angular sense). A further important detail is that in $\mathbf{H}_{\kappa_1}^2$, ultrahyperbolas for a given pair of non-intersecting focal lines are ultraellipses for the matching pair of non-intersecting focal lines (with the same end-points), hence any ultraellipse can also be understood as an ultrahyperbola, unlike ellipses and hyperbolas in $\mathbf{H}_{\kappa_1}^2$.

To make contact with the results in the physical part, we now draw our attention precisely to the conics with centre, because the orbit (24) of a particle in the oscillator potential has a centre of symmetry at the origin of the potential. In any CK space ellipses/hyperbolas have centre, as do ultraellipses/ultrahyperbolas as well as their common limits when the ‘major’ semiaxis go to infinity (if possible at all; this might happen in the hyperbolic plane, where these limiting conics are equidistants). Further analysis shows that conics appearing in the oscillator problem are generically ellipses (but not hyperbolas) and ultraellipses, with circles as the ‘equilateral’ case of an ellipse with equal semiaxes, and eventually equidistants as the non-generic limiting orbits. Parabolas have

however no centre. This means that oscillator orbits cannot be parabolas in any CK space, and henceforth these conics will be disregarded from now on.

In the sphere $\mathbf{S}_{\kappa_1}^2$ there are no either points nor lines at infinity, thus there are no limiting cases in the sense they will appear in $\mathbf{H}_{\kappa_1}^2$. This means that on the sphere there is a single type of harmonic oscillator orbits: generically ellipses on the sphere, with two limiting cases, lines through the origin (for vanishing angular momentum) and circles centred in the origin.

In the Euclidean plane \mathbf{E}^2 ultraellipses with any focal angle F are just pairs of parallel lines, in directions parallel to the two bisectors of the focal angle; in the standard description of Euclidean conics these are degenerate, and as far as harmonic oscillator trajectories these are unphysical and correspond to infinite energy and angular momentum.

The hyperbolic case $\mathbf{H}_{\kappa_1}^2$ with *negative* constant curvature is more interesting. The limiting conics relevant to the oscillator problem are obtained when the major axis of the ellipse tend to infinity, while the minor axis remains finite; the ellipse tends to a limiting conic, which turns out to be a equidistant curve to the major axis; this equidistant is also the limiting conic obtained from an ultraellipse when its ‘major axis’ tend to infinity. In \mathbf{E}^2 these limiting conics collapse precisely to pairs of parallel lines, but here they appear for finite values for E and \mathcal{J} .

4.1 Analysis of the orbits

4.1.1 The general CK configuration space

In the physical part (Section 3) we obtained, in any $\mathbf{S}_{\kappa_1[\kappa_2]}^2$, the equation of the orbit (24), which contains three free parameters. One of them, ϕ_0 , can be set to zero by taking appropriately the direction for the origin of angles. Thus the orbit equation is:

$$\frac{1}{T_{\kappa_1}^2(r)} = \frac{C_{\kappa_2}^2(\phi)}{A^2} + \frac{S_{\kappa_2}^2(\phi)}{B^2}, \quad (38)$$

which depends on two constants A and B . In the Euclidean case, this equation reduces to:

$$\frac{1}{r^2} = \frac{\cos^2(\phi)}{A^2} + \frac{\sin^2(\phi)}{B^2}, \quad (39)$$

which is an ellipse with semiaxes A, B . Again in the general CK space, the ‘physical’ constants E, E_1, E_2, \mathcal{J} are related with the parameters A, B by the relations

$$A^2 = \frac{2E_1}{\omega_0^2}, \quad B^2 = \frac{2E_2}{\omega_0^2}, \quad A^2 B^2 = \frac{\mathcal{J}^2}{\omega_0^2}. \quad (40)$$

The type of the orbit as a conic in $\mathbf{S}_{\kappa_1[\kappa_2]}^2$ depends on the space itself (the values of κ_1, κ_2) and on the constants E, E_1, E_2, \mathcal{J} , but the explicit form of the relations among E, E_1, E_2, \mathcal{J} and A, B turns out to be independent of the CK parameters κ_1, κ_2 ; for the Euclidean case $\kappa_1=0, \kappa_2=1$ they reduce to the well known expressions, and in this case the coefficients A, B in (39) are directly the orbit semiaxes. The total energy and ‘partial’ translational energy are given by

$$E = E_1 + \kappa_2 E_2 + \frac{1}{2} \kappa_1 \kappa_2 \mathcal{J}^2, \quad E_P = E_1 + \kappa_2 E_2. \quad (41)$$

The last physical constant is not the total energy, though it coincides with the total energy in all flat configuration space); it can be considered as a kind of ‘translational’ part of the energy.

4.1.2 Orbits in a Riemannian CK configuration space

We now discuss in more detail the standard Riemannian case ($\kappa_2=1$), with special emphasis in the two nonzero curvature cases. The type of the orbit depends on the values of the constants A, B . We consider altogether the family of all conics whose equation, using polar coordinates in $\mathbf{S}_\kappa^2, \mathbf{E}^2, \mathbf{H}_\kappa^2$, with the focal symmetry axis of the conic orbit as $\phi = 0$, may be written in the form:

$$\frac{1}{T_{\kappa_1}^2(r)} = \frac{\cos^2(\phi)}{A^2} + \frac{\sin^2(\phi)}{B^2}, \quad (42)$$

with $A > B > 0$. When $\kappa_1 > 0$ this equation describes spherical ellipses, with spherical circles as the particular case $A = B$ and with a spherical line (the equator) when $A = B = \infty$. When $\kappa_1 < 0$, and depending on the values of A, B , this curve which is always a conic might be generically either an ellipse or an ultraellipse in $\mathbf{H}_{\kappa_1}^2$. Before discussing each case separately, notice that in the three Riemannian cases the dependence on the polar angle ϕ is exactly the same as in the Euclidean case. The quantities A, B are related to the lengths of the two semiaxes, but as one could expect, the details depend on the sign of κ_1 : if $\kappa_1 > 0$ the range of values of $T_{\kappa_1}(r)$ is the whole real line (completed with ∞), but when $\kappa_1 < 0$, the values of $T_{\kappa_1}(r)$ are confined to the interval $[-1/\sqrt{-\kappa_1}, 1/\sqrt{-\kappa_1}]$, which reduces to $[0, 1/\sqrt{-\kappa_1}]$ for positive values for r . The values for r at $\phi = 0$ and $\phi = \pi/2$ will play the role of major a and minor b semiaxis of the conic; then the relation between the constants A, B and the geometric semiaxes is

$$T_{\kappa_1}(a) = A, \quad T_{\kappa_1\kappa_2}(b) = B. \quad (43)$$

These relations identify the geometric meaning of the constants A, B . In the non-negative curvature case, $\kappa_1 \geq 0$, any A, B will determine uniquely a, b , because the range of the circular and parabolic tangent is the whole real line. But this is not so when $\kappa_1 < 0$, for then this equation will define real values of a, b only when $A, B < 1/\sqrt{-\kappa_1}$. If both A, B are larger than $1/\sqrt{-\kappa_1}$, then the curve determined by the equation (42) is empty (as the values in the r.h.s are never got for any real value of r). Hence, we shall always assume $B < 1/\sqrt{-\kappa_1}$, so the only alternatives to study in the hyperbolic plane \mathbf{H}^2 , according to A is smaller than, equal to or larger than $1/\sqrt{-\kappa_1}$.

We remark that in the Riemannian case the origin of angles can always be chosen so that $A > B$, and the major axis is precisely the focal axis. We shall assume this choice; remark however that in the Lorentzian case $\kappa_2 < 0$ there would be two cases to be discussed, in agreement with the existence of two kinds of separation.

- (Standard) Spherical space $\kappa_1 > 0; \kappa_2 = 1$. The polar equation of the orbit is:

$$\frac{\kappa_1}{\tan^2(\sqrt{\kappa_1}r)} = \frac{\cos^2 \phi}{A^2} + \frac{\sin^2 \phi}{B^2} = \frac{\cos^2 \phi}{T_{\kappa_1}^2(a)} + \frac{\sin^2 \phi}{T_{\kappa_1}^2(b)}. \quad (44)$$

For any values for A, B with $A > B$, the values a, b (with $a > b$) are uniquely determined, and belong to the interval $[0, \pi/(2\sqrt{\kappa_1})]$. This curve is always closed, and is a spherical ellipse with centre at the potential origin. If this point is taken as the ‘pole’, the complete orbit is contained in one of the half-spheres bounded by the ‘equator’, because the r.h.s. of eq (44) is bounded by below, so the value $r = \pi/(2\sqrt{\kappa_1})$ (for which $\frac{1}{T_{\kappa_1}^2(r)} = 0$) cannot be attained. For a fixed value of \mathcal{J} the minimal value of the total energy corresponds to circles whose radius r_m satisfies $T_{\kappa_1}(r_m) = \mathcal{J}/\omega_0$ and have total energy $E_{cir} = E(\mathcal{J}) := \omega_0 \mathcal{J} + \frac{1}{2} \kappa_1 \mathcal{J}^2$. The values of the energies of the possible motions for a given \mathcal{J} lie in the interval $[E(\mathcal{J}), \infty]$; with our choice for origin of angles $E_1 > E_2$. Any oscillator orbit through any point in the upper/lower half-sphere centred at the origin is always

completely contained in that half-sphere, and from that viewpoint, there is a single limiting orbit, the ‘equator’ which corresponds to $r_m = \pi/(2\sqrt{\kappa_1})$ and $a = b = \pi/(2\sqrt{\kappa_1})$ and $A = B = \infty$; the physical constants for this orbit are $J = \infty, E = \infty$.

• (Standard) Hyperbolic space $\kappa_1 < 0; \kappa_2 = 1$. Let us now consider the general case of arbitrary negative curvature. The orbit equation is:

$$\frac{-\kappa_1}{\tanh^2(\sqrt{-\kappa_1}r)} = \frac{\cos^2 \phi}{A^2} + \frac{\sin^2 \phi}{B^2}. \quad (45)$$

This curve is always a conic in the hyperbolic plane, but its type depends on the values of A, B . This was to be expected from the discussion for the effective potential, but it is worthy to look at this situation from a more geometrical perspective.

Whitin the family of conics (45) one can expect ellipses with semiaxes a, b . These quantities should be related with A, B by means of (43). This is only possible when $A^2 < 1/(-\kappa_1), B^2 < 1/(-\kappa_1)$, for only if this condition holds the values A, B can be written as hyperbolic tangents of actual values a, b , ranging in the interval $[0, \infty]$:

$$T_{\kappa_1}^2(a) = A^2, \quad T_{\kappa_1}^2(b) = B^2.$$

Its equation in polar coordinates is:

$$\frac{-\kappa_1}{\tanh^2(\sqrt{-\kappa_1}r)} = \frac{\cos^2 \phi}{T_{\kappa_1}^2(a)} + \frac{\sin^2 \phi}{T_{\kappa_1}^2(b)}. \quad (46)$$

What happens when A^2 or B^2 lie out of the former range? If both $A^2 > 1/(-\kappa_1), B^2 > 1/(-\kappa_1)$, then for any ϕ it would follow $\tanh^2(\sqrt{-\kappa_1}r) > 1$ a condition which cannot be satisfied in any proper point of $\mathbf{H}_{\kappa_1}^2$ and this situation cannot produce any oscillator orbit. Then we are left with the case $A^2 > 1/(-\kappa_1), B^2 < 1/(-\kappa_1)$. In this case the conic (45) is *not* an hyperbolic ellipse, but a *ultraellipse*, and formally its semiaxis is not an actual distance. To cater for these cases we introduce another real quantity \tilde{a} , formally complementary to the would-be semiaxis a , and which is related to A by

$$\frac{1}{-\kappa_1 T_{\kappa_1}(\tilde{a})} = A.$$

Notice that when $\tilde{a} \in [0, \infty]$, the function $1/(\kappa_1^2 T_{\kappa_1}^2(\tilde{a}))$ ranges in the interval $[1/(-\kappa_1), \infty]$ and therefore, when considered altogether with $T_{\kappa_1}^2(a)$, the union of the ranges of the two functions $T_{\kappa_1}^2(a)$ and $1/(\kappa_1^2 T_{\kappa_1}^2(\tilde{a}))$ fills the real line. In the following we will refer to \tilde{a} as the ultraellipse ‘semiaxis’; the polar equation of the ultraellipse with semiaxes \tilde{a} and b is:

$$\frac{-\kappa_1}{\tanh^2(\sqrt{-\kappa_1}r)} = \frac{\cos^2 \phi}{\frac{1}{\kappa_1^2 T_{\kappa_1}^2(\tilde{a})}} + \frac{\sin^2 \phi}{T_{\kappa_1 \kappa_2}^2(b)} = \kappa_1^2 T_{\kappa_1}^2(\tilde{a}) \cos^2 \phi + \frac{\sin^2 \phi}{T_{\kappa_1 \kappa_2}^2(b)}. \quad (47)$$

This type of conic in $\mathbf{H}_{\kappa_1}^2$ has no generic analogue in the Euclidean plane; the Euclidean ‘limit’ $\kappa_1=0$ of (47) is a straight line, which would correspond to an Euclidean oscillator orbit with infinite energy and angular momentum. On the hyperbolic plane these oscillator orbits reach the spatial infinity with finite values for energy and angular momentum.

Therefore, in $\mathbf{H}_{\kappa_1}^2$, the equation of the complete family of conics we are considering is given, generically, by one of the two mutually exclusive possibilities:

$$\frac{-\kappa_1}{\tanh^2(\sqrt{-\kappa_1}r)} = \frac{\cos^2 \phi}{T_{\kappa_1}^2(a)} + \frac{\sin^2 \phi}{T_{\kappa_1}^2(b)}, \quad \frac{-\kappa_1}{\tanh^2(\sqrt{-\kappa_1}r)} = \kappa_1^2 T_{\kappa_1}^2(\tilde{a}) \cos^2 \phi + \frac{\sin^2 \phi}{T_{\kappa_1}^2(b)}, \quad (48)$$

where $b \in [0 \leq b < \infty]$ and either $0 \leq a < \infty$ or $\infty > \tilde{a} \geq 0$. Both types of orbits have a common limiting case, either $a \rightarrow \infty$ or $\tilde{a} \rightarrow \infty$, which corresponds to the conic

$$\frac{-\kappa_1}{\tanh^2(\sqrt{-\kappa_1}r)} = \frac{\cos^2 \phi}{\frac{1}{-\kappa_1}} + \frac{\sin^2 \phi}{T_{\kappa_1 \kappa_2}^2(b)} = (-\kappa_1)^2 \cos^2 \phi + \frac{\sin^2 \phi}{T_{\kappa_1}^2(b)}. \quad (49)$$

This conic is neither an ellipse nor an ultraellipse in the hyperbolic plane, but can be obtained from either of these types as the limit $a \rightarrow \infty$ or $\tilde{a} \rightarrow \infty$ while b is fixed. In the Euclidean case, the limiting ellipse is clearly a straight line, equidistant to the major axis and with b as equidistance. In the hyperbolic plane, this conic is an equidistant curve, with equidistance b from the x -axis. This means that in the case of the hyperbolic plane $\mathbf{H}_{\kappa_1}^2$ the family of conics we are considering includes conics intersecting the major axis at proper points (the ellipse), at only at infinity (the equidistant) or not intersecting it at all (the ultraellipse).

Of course the two expressions (48) can be used as well when $\kappa_1 > 0$ but then each of the two alternatives covers actually all cases. And further, only the first possibility has a sensible Euclidean limit because the $\kappa_1 \rightarrow 0$ limit of the \tilde{a} family lies completely at the infinity; the existence of this family as a different set of conics is specific to the hyperbolic plane.

In Section 3 we discussed the appearance of some energy and angular momentum standards $E_\infty, \mathcal{J}_\infty$ in the hyperbolic plane. As it might be expected, these correspond to the transition among different types of oscillator orbits. To be precise, very simple calculations show that in \mathbf{H}^2 :

- All elliptic orbits have energies $E < E_\infty$ and angular momentum $\mathcal{J} < \mathcal{J}_\infty$
- The equidistant orbits have energy $E = E_\infty$ and angular momentum $\mathcal{J} < \mathcal{J}_\infty$
- The ultraelliptic orbits have energies $E > E_\infty$ and the angular momentum \mathcal{J} can have any value. The ultraellipses with $\mathcal{J} < \mathcal{J}_\infty$ are those with $T_{\kappa_1}(b) < T_{\kappa_1}(\tilde{a})$, while those with $T_{\kappa_1}(b) > T_{\kappa_1}(\tilde{a})$ have $\mathcal{J} > \mathcal{J}_\infty$. Separating these behaviours are the ‘equilateral’ ultraellipses with $T_{\kappa_1}(b) = T_{\kappa_1}(\tilde{a})$, all of which have $\mathcal{J} = \mathcal{J}_\infty$.
- The ‘largest circular orbit’, with radius $r = \infty$ is a common limit from elliptic orbits when $a \rightarrow \infty, b \rightarrow \infty$ or from equidistant orbits, when $b \rightarrow \infty$. This orbit has energy $E = E_\infty$ and angular momentum $\mathcal{J} = \mathcal{J}_\infty$

It is clear that when the total energy E is smaller than E_∞ , the trajectory (an ellipse) is bounded and the motion is periodic, while for E equal to or larger than E_∞ , the motion is not periodic and the orbit goes to spatial infinity.

We finally give the relation between the two ‘geometric’ quantities, the ellipse (or ultraellipse) semiaxes a (or \tilde{a}) and b , the parameters A, B appearing in the canonical form of the orbit (38) and the angular momentum \mathcal{J} and energies E, E_1, E_2 :

$$\omega_0^2 T_{\kappa_1}^2(a) = \omega_0^2 A^2 \equiv 2E_1, \quad \text{or} \quad \omega_0^2 \frac{1}{\kappa_1^2 T_{\kappa_1}^2(\tilde{a})} = \omega_0^2 A^2 \equiv 2E_1, \quad (50)$$

$$\omega_0^2 T_{\kappa_1 \kappa_2}^2(b) = \omega_0^2 B^2 \equiv 2E_2, \quad (51)$$

$$\mathcal{J} = \omega_0 T_{\kappa_1}(a) T_{\kappa_1 \kappa_2}(b), \quad \text{or} \quad \mathcal{J} = \omega_0 \frac{1}{-\kappa_1 T_{\kappa_1}(\tilde{a})} T_{\kappa_1 \kappa_2}(b), \quad (52)$$

$$E = \frac{1}{2} \omega_0^2 \left\{ \frac{T_{\kappa_1}^2(a)}{C_{\kappa_1}^2(b)} + \kappa_2 T_{\kappa_1 \kappa_2}^2(b) \right\} \quad \text{or} \quad E = \frac{1}{2} \omega_0^2 \left\{ \frac{1}{\kappa_1^2 T_{\kappa_1}^2(\tilde{a}) C_{\kappa_1}^2(b)} + \kappa_2 T_{\kappa_1 \kappa_2}^2(b) \right\}. \quad (53)$$

5 Final comments and outlook

We have discussed and completely solved the harmonic oscillator problem simultaneously on the nine 2d spaces of constant curvature and metric of either signature type. As stated in the introduction, one of the fundamental characteristics of this approach is the use of two parameters κ_1, κ_2 in such a way that all properties we have obtained reduce to the appropriate property for the system on each particular space. Generically these spaces include either the sphere \mathbf{S}^2 , the hyperbolic plane \mathbf{H}^2 , the De Sitter \mathbf{dS}^{1+1} or the AntiDeSitter spaces \mathbf{AdS}^{1+1} when the corresponding values of κ_1, κ_2 are appropriately set. Euclidean or Minkowskian spaces arise as the very particular (but important) flat cases $\kappa_1 = 0$ with a non-degenerate metric $\kappa_2 \neq 0$. So, we can sum up the results pointing out some important facts:

- The harmonic oscillator is not a specific or special system living only on the Euclidean space. In any of the nine CK spaces $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ there is a system with all the outstanding properties of the Euclidean harmonic oscillator, worth of the name ‘curved harmonic oscillator’.
- Motion in the ‘curved harmonic oscillator’ potential on any CK space is superintegrable.

The orbits of motion in this potential are always conics with centre at the origin of potential, in all the CK spaces and in the sense of the intrinsic geometry in $\mathcal{S}_{\kappa_1[\kappa_2]}^2$. These conics always include ellipses, with straight lines trough the origin as limiting cases for $\mathcal{J} = 0$, but other types of conics might also appear in some spaces. For the three Riemannian spaces of constant curvature, the situation is as follows:

- In the sphere case, there is a single type of oscillator orbits; all such orbits are ellipses and are confined to an hemisphere centred in the oscillator centre; a ‘limiting’ case is the ‘equator’ orbit, which appears as an ellipse with major axis equal to half the equator’s length (on the sphere this condition implies both axis to be equal); this orbit has infinite energy and angular momentum.
- In the Euclidean case (the $\kappa_1 \rightarrow 0$ limit of the sphere case), all oscillator orbits are ellipses, and a limiting case happens when the orbit tends to a straight line not through the origin, with infinite angular momentum and energy.
- In the hyperbolic plane, orbits with values of E smaller than E_∞ are ellipses, and for them \mathcal{J} should be smaller than \mathcal{J}_∞ . This ellipse reaches the spatial infinity when $E = E_\infty$; this happens when the focal distance goes to infinity but the minor axis stays finite, and the limiting curve is an equidistant, with base curve the major axis and equidistance distance equal to the minor semiaxis. For values of $E > E_\infty$ the orbit is (a branch of) an ultraellipse, a conic reaching the spatial infinity, and in this case all values of \mathcal{J} (even larger than \mathcal{J}_∞) are allowed; in the limit where E, \mathcal{J} go to infinity, the orbit tends to a hyperbolic straight line, which appears here as a particular limiting case of an ultraellipse.

Then, as compared with the Euclidean case, the new trait appearing in the hyperbolic plane $\mathbf{H}_{\kappa_1}^2$ is the ‘splitting’ of the Euclidean ‘last’ singular straight line in a full family of orbits: a ‘first’ orbit reaching the spatial infinity, then a full family of ultraelliptic orbits and finally a straight line orbit (Figure 4). And the new trait in the sphere case is that the Euclidean ‘last’ singular orbit becomes the sphere equator, which is also a straight line in the spherical geometry. Qualitatively, this reminds the results in the Kepler problem, where the single Euclidean parabolic orbit ‘splits’ for $\kappa_1 < 0$ into a full family of Kepler orbits in \mathbf{H}^2 , bordered by two different limiting curves, an

horoellipse as a limiting form of ellipses and a horohyperbola as a limiting form of hyperbolas, with a full interval of parabolas between them. The separating role in the oscillator problem is played by equidistants.

Of course, since the nine CK manifolds $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ are geometrically very different, many dynamical properties display some differences according to some distinguishing properties of the manifolds; nevertheless, all these differences can be traced back ultimately to characteristics related to the signs of the basic parameters κ_1, κ_2 so there is a unique theory that is simultaneously valid for all the cases, for any value of the curvature and any signature type.

The analysis of the orbits in the curved harmonic oscillator leads, in a natural way, to the theory of conics on spaces of constant curvature and any signature type. Although Sects. 3 and 4 were mainly concerned with dynamical questions, Sect. 5 was written emphasizing its geometrical character. It is clear that the theory of conics on spaces of constant curvature is a geometrical matter of some importance deserving a deeper study within this CK formalism that we hope to present elsewhere; in the three Riemannian spaces of constant curvature conics have been discussed, at different depths and from differing viewpoints, in papers dating from a century or more (see for instance [36], p. 229 or [15]). Of course, none of these papers discussed the case of Lorentzian spaces, where it appears that this theory has been never presented systematically.

Appendix: Geodesic coordinates on two-dimensional manifolds

Consider the generic CK space $\mathcal{S}_{\kappa_1[\kappa_2]}^2$, for any values of κ_1, κ_2 . When κ_2 is positive it may be reduced to 1, and then this family includes the three constant curvature 2d Riemannian spaces $V_{\kappa_1}^2$. When κ_2 is negative, it may be reduced to -1 and the spaces are Lorentzian manifolds of constant curvature $L_{\kappa_1}^2$. We now describe the two types of coordinates employed in the paper. Choose any point O a point on $\mathcal{S}_{\kappa_1[\kappa_2]}^2$ and let l_1 be an oriented geodesic (time-like if κ_2 is non-positive, generated by P_1) through O , and l_2 the oriented geodesic orthogonal to l_1 through O (hence space-like if κ_2 is non-positive and generated by P_2) (see Figs. 5 and 6)

I: Geodesic polar coordinates

For any point Q in some suitable domain (with time-like separation to O in the lorentzian case), there is a unique geodesic l joining Q with O . The (geodesic) polar coordinates (r, ϕ) of Q , relative to the origin O and the positive geodesic ray of l_1 , are the distance r between Q and O measured along l , and the angle ϕ between l and the positive ray l_1 , measured around O (Figure 5). These coordinates are defined in some domain not extending beyond the cut locus of O , are singular at O ; when $\kappa_2 > 0$, ϕ is discontinuous on the positive ray of l_1 , where there is a jump of $2\pi/\sqrt{\kappa_1}$, but if $\kappa_2 < 0$ then the range of ϕ covers all real line, and there are no jumps, at the price of the coordinates being singular on the isotropes through O . In the sphere the domain of polar coordinates only fails to include two points: O and its antipodal; for the hyperbolic plane it covers all the space except the point O ; for the Anti De Sitter this covers the domain with time-like separation to O , save O and its antipodal, etc.

The expression for the differential element of distance dl is given by

$$ds^2 = dr^2 + \kappa_2 S_{\kappa_1}^2(r) d\phi^2,$$

so that in the standard Euclidean case ($\kappa_1=0, \kappa_2=1$) we get $ds^2|_{\mathbb{E}^2} = dr^2 + r^2 d\phi^2$.

II: Geodesic parallel coordinates

For any point Q in some suitable domain, there is a unique geodesic l'_2 through Q and orthogonal to l_1 , intersecting l_1 at a point denoted Q_1 ; the geodesic l'_2 is space-like in the Lorentzian case. Alternatively, for Q in some suitable domain, there is a unique geodesic l'_1 through Q and orthogonal to l_2 intersecting l_2 at a point denoted Q_2 ; the geodesic l'_1 is time-like in the Lorentzian case. The points Q_1, Q_2 can be considered as the ‘orthogonal projections’ of Q on the lines l_1, l_2 (recall that $S^2_{\kappa_1[\kappa_2]}$ is in general not an affine space). Then we can characterize the point Q by (Figure 6)

1. Two coordinates (u, y) . The coordinate u is the canonical parameter of the element in the one-dimensional subgroup of translations along l_1 , generated by P_1 and with label κ_1 , which brings O to Q_1 ; this value coincides with the distance between O and Q_1 computed along l_1 with the CK space metric. The coordinate y is the canonical parameter of the element in the one-dimensional subgroup of translations along l'_2 , generated by $e^{uP_1} P_2 e^{-uP_1}$ and with label $\kappa_1 \kappa_2$, which brings Q_1 to Q ; this value is related with the distance between Q_1 and Q computed along l'_1 with the CK space metric by a factor $\sqrt{\kappa_2}$ (remark in the Lorentzian case, y is always real, yet both a space-like separation and $\sqrt{\kappa_2}$ are pure imaginary).
2. Two coordinates (x, v) . The coordinate v is the canonical parameter of the element in the one-dimensional subgroup of translations along l_2 , generated by P_2 and with label $\kappa_1 \kappa_2$, which brings O to Q_2 ; this value is related with the distance between O and Q_2 computed along l_2 with the CK space metric by a factor $\sqrt{\kappa_2}$, so that v is always real even in the Lorentzian case. The coordinate x is the canonical parameter of the element in the one-dimensional subgroup of translations along l'_1 , generated by $e^{vP_2} P_1 e^{-vP_2}$ and with label κ_1 , which brings Q_2 to Q ; this value coincides with the distance between Q_2 and Q computed along l'_1 with the CK space metric.

In the first case we have the parallel coordinates of Q relative to (O, l_1) and in the second case relative to (O, l_2) (Figure 6). In the (u, y) system the curves ‘ $u = \text{constant}$ ’ are geodesics meeting orthogonally the ‘base’ geodesic l_1 , and the curves ‘ $y = \text{constant}$ ’ are equidistant lines to the base l_1 , and intersect orthogonally $u = \text{constant}$. In the (x, v) system the curves ‘ $v = \text{constant}$ ’ are geodesics and the lines ‘ $x = \text{constant}$ ’ are equidistant to l_2 . Notice that in the general case, with non-zero curvature $\kappa_1 \neq 0$ we have $u \neq x$ and $v \neq y$; only in the case of flat spaces do the equalities $x = u, v = y$ hold.

The (u, y) and (x, v) expressions for the differential element of distance ds^2 are given by [37]

$$ds^2 = C_{\kappa_1 \kappa_2}^2(y) du^2 + \kappa_2 dy^2, \quad ds^2 = dx^2 + \kappa_2 C_{\kappa_1}^2(x) dv^2,$$

The three coordinate systems can be related by the general formulae of trigonometry in the CK space $S^2_{\kappa_1[\kappa_2]}$ [23].

$$T_{\kappa_1}(u) = T_{\kappa_1}(r) C_{\kappa_2}(\phi), \quad S_{\kappa_1 \kappa_2}(y) = S_{\kappa_1}(r) S_{\kappa_2}(\phi), \quad C_{\kappa_1}(u) C_{\kappa_1 \kappa_2}(y) = C_{\kappa_1}(r).$$

In particular, the two parallel coordinate systems coincide when $\kappa_1 = 0$ reducing to $ds^2 = dx^2 + \kappa_2 dy^2$; in the standard Euclidean case with $\kappa_2 = 1$ this reduces further to $ds^2 = dx^2 + dy^2$. In this article we have made use of only the (u, y) coordinates, but all the CK formulation can be easily expressed also in the (x, v) coordinates.

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References

- [1] R. ABRAHAM, J.E. MARSDEN AND T. RATIU, *Manifolds, tensor analysis, and applications*, 2nd ed., Applied mathematical Sciences (Springer-Verlag, New York, 1988).
- [2] A. BALLESTEROS, F.J. HERRANZ, M.A. DEL OLMO AND M. SANTANDER, Quantum structure of the motion groups of the two-dimensional Cayley-Klein geometries, *J. Phys. A* **26**, 5801–5823 (1993).
- [3] A. BALLESTEROS, F.J. HERRANZ, M. SANTANDER AND T. SANZ-GIL, Maximal superintegrability on N -dimensional curved spaces, *J. Phys. A* **36**, L93–99 (2003).
- [4] A.O. BARUT AND R. WILSON, On the dynamical group of the Kepler problem in a curved space of constant curvature, *Phys. Lett. A* **110**, 351–354 (1985).
- [5] A.O. BARUT, A. INOMATA AND G. JUNKER, Path integral treatment of the hydrogen atom in a curved space of constant curvature, *J. Phys. A* **20**, 6271–6280 (1987).
- [6] A.O. BARUT, A. INOMATA AND G. JUNKER, Path integral treatment of the hydrogen atom in a curved space of constant curvature: II, *J. Phys. A* **23**, 1179–1190 (1990).
- [7] I.M. BENN, Geodesics and Killing Tensors in Mechanics, *J. Math. Phys.* **47**, 022903 (15) (2006).
- [8] G. BIRKHOFF, Confocal conics in space-time, *Am. Math. Monthly* **69** 1–4 (1962).
- [9] D. BONATSOS, C. DASKALOYANNIS AND K. KOKKOTAS, Quantum-algebraic description of quantum superintegrable systems in two dimensions, *Phys. Rev. A* **48**, R3407–R3410 (1993).
- [10] D. BONATSOS, C. DASKALOYANNIS AND K. KOKKOTAS, Deformed oscillator algebras for two-dimensional quantum superintegrable systems, *Phys. Rev. A* **50**, 3700–3709 (1994).
- [11] J.F. CARIÑENA, M.F. RAÑADA, M. SANTANDER AND T. SANZ-GIL, Separable potentials and triality in 2d-spaces of constant curvature, *J. Nonlinear Math. Phys.* **12**, 230–252 (2005).
- [12] J.F. CARIÑENA, M.F. RAÑADA AND M. SANTANDER, Central potentials on spaces of constant curvature: The Kepler problem on the two-dimensional sphere S^2 and the hyperbolic plane H^2 , *J. Math. Phys.* **46**, 052702, 1–25 (2005).
- [13] J.F. CARIÑENA, M.F. RAÑADA AND M. SANTANDER, The Kepler problem and the Laplace-Runge-Lenz vector on spaces of constant curvature and arbitrary signature, *Qualitative theory of dynamical systems* to appear (2007).
- [14] M. DO CARMO, *Riemannian geometry*, Birkhäuser Boston, Boston, MA, 1992.
- [15] J.L. COOLIDGE, *The elements of non-Euclidean geometry*, Oxford at Clarendon Press, 1909. Chapter XII.
- [16] B. CORDANI, *The Kepler problem. Group theoretical aspects, regularization and quantization, with application to the study of perturbations*, Progress in Mathematical Physics, 29. Birkhäuser Verlag, Basel, (2003).
- [17] P. DOMBROWSKI AND J. ZITTERBARTH, On the planetary motion in the 3-Dim standard spaces M_κ^3 of constant curvature κ , *Demonstratio Mathematica* **24**, 375–458 (1991).
- [18] D.M. FRADKIN, Three-dimensional isotropic harmonic oscillator and $SU(3)$, *Am. J. Phys.* **33**, 207–211 (1965).
- [19] H. GOLDSTEIN, *Classical mechanics*, Addison-Wesley Publishing Co., Reading, Mass. (Second edition, 1980).
- [20] C. GROSCHE, The path integral for the Kepler problem on the pseudosphere, *Ann. Phys.* **204**, 208–222 (1990).
- [21] F.J. HERRANZ AND M. SANTANDER, The general solution of the real Z_2^N graded contractions of $so(N+1)$ *J. Phys. A* **29**, 6643–6652 (1996).
- [22] F.J. HERRANZ AND M. SANTANDER, Casimir Invariants for the complete family of quasi-simple orthogonal algebras *J. Phys. A* **30**, 5411–5426 (1997).

- [23] F.J. HERRANZ, R. ORTEGA AND M. SANTANDER, Trigonometry of space-times: a new self-dual approach to a curvature/signature (in)dependent trigonometry, *J. Phys. A* **33**, 4525–4551 (2000).
- [24] F.J. HERRANZ AND M. SANTANDER, Conformal symmetries of space-times, *J. Phys. A* **35**, 6601–6618 (2002).
- [25] F.J. HERRANZ AND M. SANTANDER, Conformal compactification of space-times, *J. Phys. A* **35**, 6619–6629 (2002).
- [26] F.J. HERRANZ, A. BALLESTEROS, M. SANTANDER AND T. SANZ-GIL, Maximally superintegrable Smorodinsky-Winternitz systems on the N -dimensional sphere and hyperbolic spaces, in Proc. of *Superintegrability in classical and quantum systems*, edited by P. Tempesta, P. Winternitz et al, CRM Proc. Lecture Notes, 37, pp. 75–89, Amer. Math. Soc., Providence, RI (2004).
- [27] F.J. HERRANZ AND A. BALLESTEROS, Superintegrability on Three-Dimensional Riemannian and Relativistic Spaces of Constant Curvature, *SIGMA (Symmetry, Integrability and Geometry: Methods and Applications* **2**, 010, 22p (2006). Available online at <http://www.emis.de/journals/SIGMA/2006/Paper010/>
- [28] P.W. HIGGS, Dynamical symmetries in a spherical geometry I, *J. Phys. A* **12**, 309–323 (1979).
- [29] J.M. JAUCH AND E.L. HILL, On the problem of degeneracy in quantum mechanics, *Phys. Rev.* **57**, 641–645 (1940).
- [30] E.G. KALNINS, W. MILLER AND G.S. POGOSYAN Superintegrability on the two-dimensional hyperboloid, *J. Math. Phys.* **38**, 5416–5433 (1997).
- [31] E.G. KALNINS, W. MILLER AND G.S. POGOSYAN, Coulomb-oscillator duality in spaces of constant curvature, *J. Math. Phys.* **41**, 2629–2657 (2000).
- [32] E.G. KALNINS, J.M. KRESS, G.S. POGOSYAN AND W. MILLER, Completeness of superintegrability in two-dimensional constant-curvature spaces, *J. Phys. A* **34**, 4705–4720 (2001).
- [33] E.G. KALNINS, W. MILLER AND G.S. POGOSYAN, The Coulomb-oscillator relation on n -dimensional spheres and hyperboloids, *Phys. of Atomic Nuclei* **65**, 1119–1127 (2002).
- [34] E.G. KALNINS, J.M. KRESS AND W. MILLER, Second-order superintegrable systems in conformally flat spaces. I. Two-dimensional classical structure theory, *J. Math. Phys.* **46**, 053509, 1–28 (2005).
- [35] E.G. KALNINS, J.M. KRESS AND W. MILLER, Second order superintegrable systems in conformally flat spaces. II. The classical two-dimensional Stäckel transform, *J. Math. Phys.* **46**, 053510, 1–15 (2005).
- [36] F. KLEIN, *Vorlesungen über nicht-euklidische Geometrie* (Die Grundlehren der mathematischen Wissenschaften in der Einzeldarstellungen, Verlag von Julius Springer, Berlin, 1928).
- [37] W. KLINBERGER, *A course in differential geometry* (Graduate texts in Mathematics, Springer-Verlag, New York, 1978).
- [38] H.I. LEEMON, Dynamical symmetries in a spherical geometry II, *J. Phys. A* **12**, 489–501 (1979).
- [39] H. LIEBMANN, *Nichteuklidische Geometrie*, 1st ed. (Göschen'sch Verlag, Leipzig, 1905) ; 3rd ed. (De Gruyter, Berlin, Leipzig, 1923).
- [40] C. LÓPEZ, E. MARTÍNEZ AND M.F. RAÑADA, Dynamical symmetries, non-Cartan symmetries, and superintegrability of the n -dimensional Harmonic Oscillator, *J. Phys. A* **32**, 1241–1249 (1999).
- [41] G. MARMO, E. SALETAN, A. SIMONI AND B. VITALE, *Dynamical Systems: A Differential Geometric Approach to Symmetry and Reduction*, (Wiley, Chichester, 1985).
- [42] R.G. MCLENAGHAN AND R.G. SMIRNOV, Intrinsic Characterization of Orthogonal Separability for Natural Hamiltonians with Scalar potentials on Pseudo-Riemannian Spaces, *J. Nonlinear Math. Phys.* **9**, 140–151 (2002).
- [43] R.G. MCLENAGHAN, R.G. SMIRNOV AND D. THE, An extension of the classical theory of algebraic invariants to pseudo-Riemannian geometry and Hamiltonian mechanics, *J. Math. Phys.* **45**, 1079–1120 (2004).
- [44] R. ORTEGA AND M. SANTANDER, Trigonometry of complex Hermitian-type homogeneous symmetric spaces, *J. Phys. A* **35**, 7877–7917 (2002).
- [45] M.F. RAÑADA, Superintegrable $n = 2$ systems, quadratic constants of motion, and potentials of Drach, *J. Math. Phys.* **38**, 4165–4178 (1997).
- [46] M.F. RAÑADA, Dynamical symmetries, bi-Hamiltonian structures, and superintegrable $n = 2$ systems, *J. Math. Phys.* **41**, 2121–2134 (2000).

- [47] M.F. RAÑADA AND M. SANTANDER, Superintegrable systems on the two-dimensional sphere S^2 and the hyperbolic plane H^2 , *J. Math. Phys.* **40**, 5026–5057 (1999).
- [48] M.F. RAÑADA AND M. SANTANDER, Complex Euclidean superintegrable potentials, potentials of Drach, and potential of Holt, *Phys. Lett. A* **278**, 271–279 (2001).
- [49] M.F. RAÑADA AND M. SANTANDER, On some properties of harmonic oscillator on spaces of constant curvature, *Rep. Math. Phys.* **49**, 335–343, (2002).
- [50] M.F. RAÑADA AND M. SANTANDER, On the Harmonic Oscillator on the two-dimensional sphere S^2 and the hyperbolic plane H^2 , *J. Math. Phys.* **43**, 431–451 (2002).
- [51] M.F. RAÑADA AND M. SANTANDER, On the Harmonic Oscillator on the two-dimensional sphere S^2 and the hyperbolic plane H^2 II, *J. Math. Phys.* **44**, 2149–2167 (2003).
- [52] M.F. RAÑADA, T. SANZ-GIL AND M. SANTANDER, Superintegrable potentials and superposition of Higgs oscillators on the sphere S^2 , in *Classical and Quantum integrability (Warsaw 2001)*, Banach Center Publications Vol. **59**, 243–255, Acad. Sci. Warsaw (2003).
- [53] M.F. RAÑADA AND M. SANTANDER, On the Superintegrability of a rational oscillator with inversely quadratic nonlinearities: Euclidean and non-Euclidean cases, in *Proc. of Superintegrability in classical and quantum systems*, edited by P. Tempesta, P. Winternitz et al, CRM Proc. Lecture Notes, **37**, pp. 161–172, Amer. Math. Soc., Providence, RI (2004).
- [54] M. SANTANDER, The Hyperbolic-AntiDeSitter-DeSitter triality, in *Proceedings of the Meeting Lorentzian Geometry* (Benalmádena, Spain), Publications of the RSME, **5**, 247–260, Cañadas-Pinedo M.A., Gutiérrez M. and Romero A. Eds. (2003).
- [55] M. SANTANDER AND T. SANZ-GIL, From oscillator(s) and Kepler(s) potentials to general superintegrable systems in spaces of constant curvature, *Rep. on Math. Phys.*, **55**, pp. 371–383, (2005).
- [56] E. SCHROEDINGER, A method of determining quantum mechanical eigenvalues and eigenfunctions, *Proc. R.I.A.* **A 46**, 9–16 (1940).
- [57] J.J. SLAWIANOWSKI, Bertrand systems on spaces of constant sectional curvature, *Rep. Math. Phys.* **46**, 429–460 (2000).
- [58] W.E. STORY, On non-Euclidean properties of conics, *Amer. J. Math.*, Baltimore, Md., **5**, pp. 358–381, (1882).
- [59] M. S. SZYDŁOWSKI, M. HELLER AND W. SASIN, Geometry of spaces with the Jacobi metric *J. Math. Phys.* **37**, 346–360 (1996).
- [60] P. TEMPESTA, A.V. TURBINER AND P. WINTERNITZ, Exact solvability of superintegrable systems, *J. Math. Phys.* **42**, 4248–4257 (2001).
- [61] T.G. VOZMISCHEVA, *Integrable problems of celestial mechanics in spaces of constant curvature*, Astrophysics and Space Science Library, 295. (Kluwer Academic Pub., Dordrecht, 2003).
- [62] E.T. WHITTAKER, *A treatise on the Analytical Dynamics of particles and rigid bodies*, Cambridge Univ. Press (Fourth edition, 1965).

Figures and figure captions

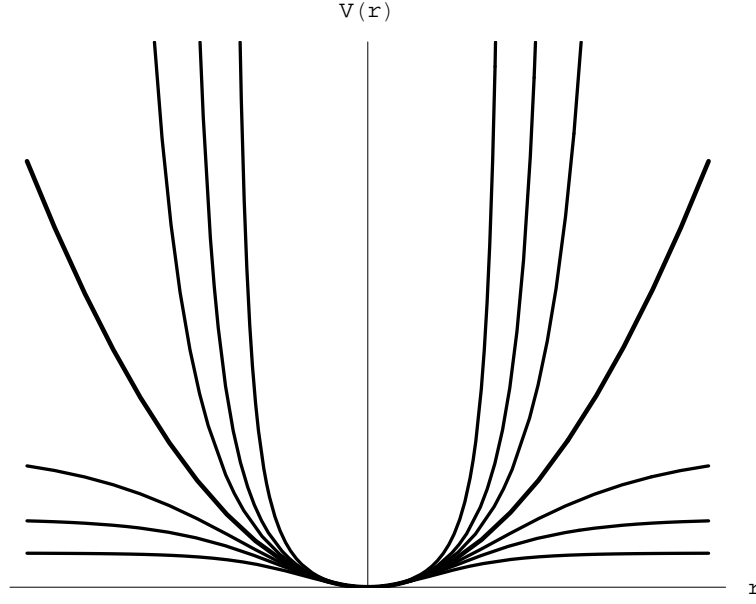


FIGURE 1. Plot of the harmonic oscillator Potential $\mathcal{V}(r)$ as a function of r , for several values for the curvature. Upper curves correspond to positive curvature $\kappa = 2, 1, 0.5$; the slightly thicker line to the Euclidean plane $\kappa = 0$; the lower curves to negative curvatures $\kappa = -0.5, -1, -2$. All the functions have the same quadratic behaviour around $r = 0$ and the quadratic Euclidean function appears in this formalism as making a separation between two different behaviours (an infinite wall at finite r versus a finite plateau at infinite r). In the case $\kappa_2 > 0$ the natural range of the radial coordinate includes only positive values, but when $\kappa_2 < 0$ positive and negative r appear naturally.

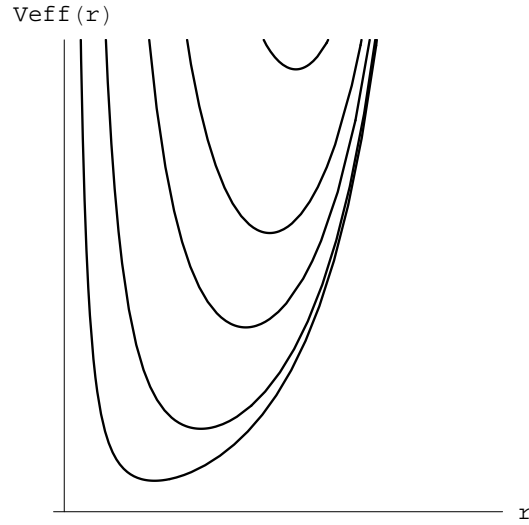


FIGURE 2. Plot of the effective potential $\mathcal{V}^{\text{eff}}(r)$ as a function of r in the standard Riemannian positive curvature case ($\kappa_1 = 1, \kappa_2 = 1$) depicted for several values for \mathcal{J} . All these potentials are asymmetric wells with two infinite walls at $r = 0$ and $r = \pi/2$.

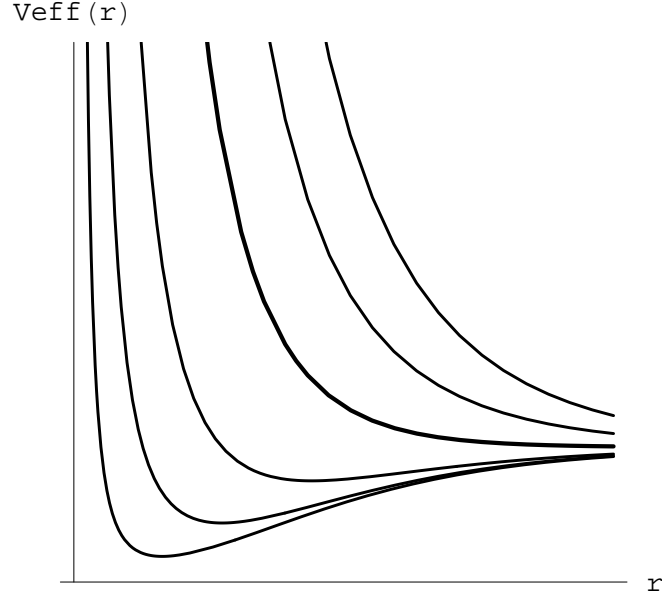


FIGURE 3. Plot of the effective potential $\mathcal{V}^{\text{eff}}(r)$ as a function of r in the standard Riemannian negative curvature case ($\kappa_1 = -1, \kappa_2 = 1$) depicted for several values for \mathcal{J} . The angular momentum standard \mathcal{J}_∞ corresponds to the slightly thicker curve, where behaviour of the effective potential changes. Curves for values of \mathcal{J} greater (resp. lower) than \mathcal{J}_∞ appear above (resp. below) this curve and correspond to an equivalent potential without (resp. with) a minimum.

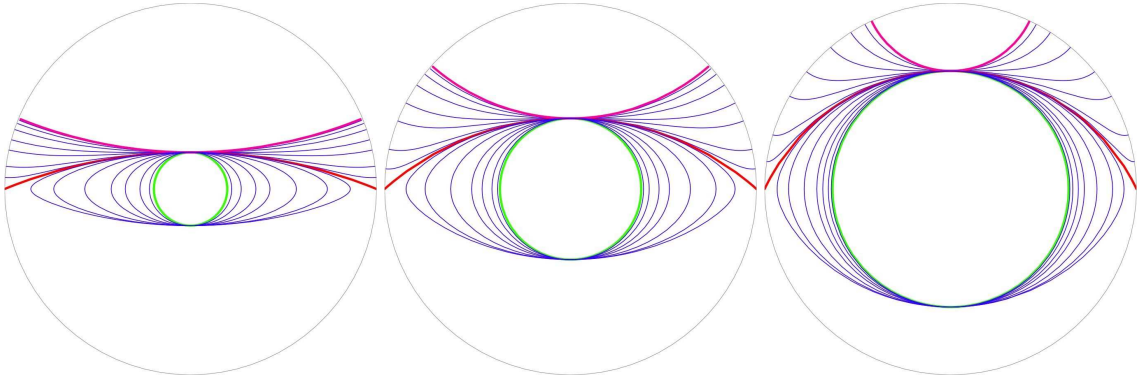


FIGURE 4 ABC. Harmonic oscillator orbits in a hyperbolic plane configuration space, depicted in the conformal Poincaré disk model. Each figure displays orbits with a fixed value for the minor semiaxis b (or equivalently, fixed ‘partial energy’ E_2) and several values for the major semiaxis a , ranging from $a = b$ (circular orbit, in green), seven ellipses for increasing values of a (in blue), an equidistant curve for $a = \infty$ or $\tilde{a} = \infty$ (in red), seven ultraellipses for decreasing values of \tilde{a} (in blue) and finally the straight orbit for $\tilde{a} = 0$ (in magenta). From left to right, b is ranging from ‘small’ (Figure 4a) to ‘large’ (Figure 4c) values. The potential centre is at the origin, which is a centre of the conics. Colors have been chosen to represent particular and limiting conics: circle (green), equidistant (red) and straight line (magenta). For ellipses the pair of focus (not marked) are on the horizontal line; for the equidistant the foci are at infinity, as well the focal lines, which are orthogonal to the horizontal line l_1 at infinity; for the ultraellipses one set of focal lines is orthogonal to the horizontal line. Notice only orbits with total energy smaller than E_∞ intersect the horizontal line l_1 and come back to the initial point. Orbits with total energy larger than this value are not closed and go to spatial infinity. The two families in blue (ellipses and ultraellipses) are the two generic behaviours, as explained in the text.

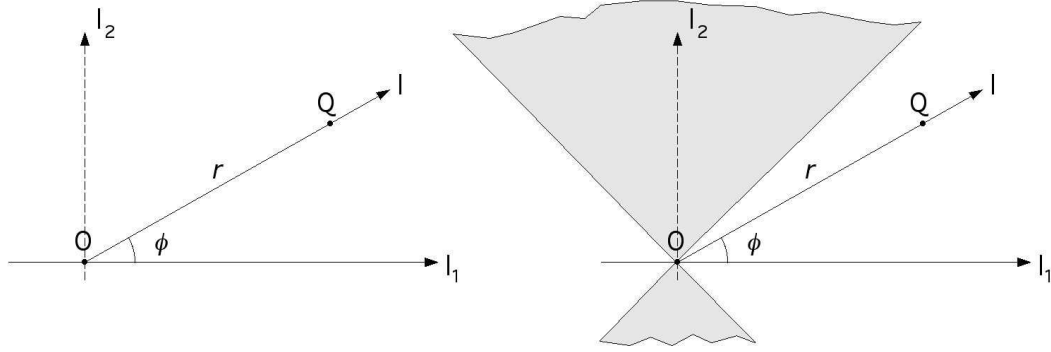


FIGURE 5 AB. The ‘polar’ coordinates (r, ϕ) . The diagram depicts the geometrical meaning of polar coordinates (r, ϕ) in a general CK space $S^2_{\kappa_1[\kappa_2]}$, both in the locally Riemannian case $\kappa_2 > 0$ (left) and in the pseudo-Riemannian case $\kappa_2 < 0$ (right). In all cases l_1, l_2, l are geodesics, and l_1, l_2 are orthogonal. The light cone through O is also shown in the lorentzian diagram. The coordinate r has label κ_1 while ϕ has label κ_2 . In the Riemannian case, the coordinate r is non-negative, only vanishes at point O , where polar coordinates are singular, and the angular coordinate ϕ ranges in the interval $[0, 2\pi/\sqrt{\kappa_2}]$ with the usual periodic conditions. In the pseudo-Riemannian case r vanishes along the isotropes through O (and would be pure imaginary in the shaded area with space-like separation to O); the angle ϕ ranges in the interval $[-\infty, \infty]$ and for a given ϕ the natural range of r involves positive as well as negative values.

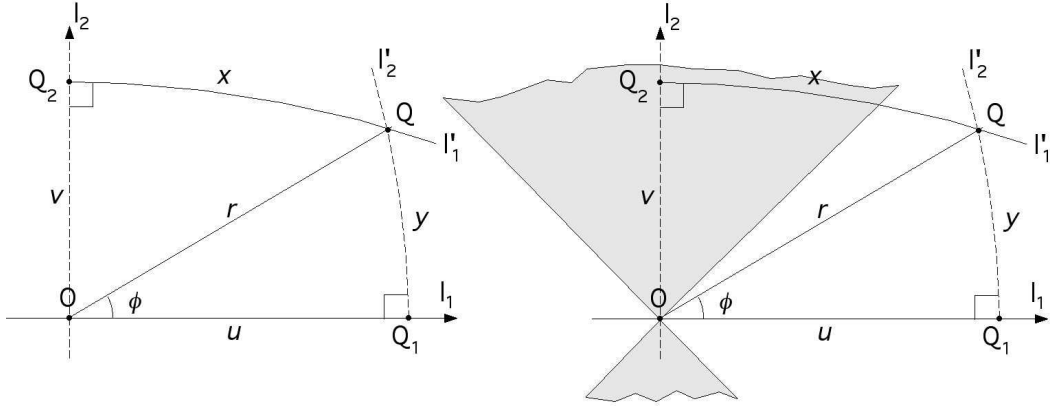


FIGURE 6 AB. The ‘parallel’ coordinates (u, y) and (v, x) . The diagram depicts the geometrical meaning of the coordinates (u, y) and (v, x) , for the same situation and with the same conventions as in Fig.5. The lines l'_1, l'_2 are geodesics through Q orthogonal to l_2, l_1 respectively. The coordinates u, x have label κ_1 and are (locally) defined near O in both the Riemannian and pseudo-Riemannian cases. The coordinates v, y have label $\kappa_1\kappa_2$ and the corresponding geodesics are represented dashed; in the pseudo-Riemannian case this means these geodesics are space-like. In all cases the ordinary sign convention applies. When $\kappa_1 \neq 0$, $x \neq u$ and $v \neq y$, and equality is a degenerate property of the flat case. See the text in the appendix for more details, and note that the natural interpretation of all coordinates is as canonical parameters of one-parameter subgroup of translations along the lines l_1, l_2, l'_1, l'_2 or of rotations around the point O .